This is your final (take-home) examination for STAT 464/864. Undergraduates are expected to complete the first 5 questions, and graduate students the full exam. You should note that for the undergrads, there are only 35 marks, so each mark corresponds to a percent in your final grade. For the graduate students, there are 2 marks per final percent: expect the grading to be correspondingly harder.

Since this is a take-home, and you have a lot of time to do it, I will expect complete, thorough, carefully written solutions. You are on the honor system with regards to discussion with classmates or others, but are welcome to use any textual resources you want (including all of the course reserve texts, and anything else you want from the library).

The exam is due to me **by** 3pm, December 21st. No late submissions will be accepted. Failure to hand-in the exam will result in a failing grade in the class. Don't sleep in!

Good luck!

1. [10 marks] (R problem) Using `arima.sim()`, generate an ARMA(p,q) time series ($p, q \geq 2$ your choice, and ϕ,θ also your choice) of length 1000 points. Taking the simulated values, fit an ARMA(p',q') model to it, where you select p' and q' using an iterative process similar to your reports. That is, you should not assume you know anything about the process, and explore a variety of fits and decide upon the best one using a logical process. Hand in a succinct set of code and results – marks will be taken off for too much verbosity. Feel free to use carefully written comments to explain your rationale in the code, or alternatively hand in a short write-up that explains your thought processes.

2. [5 marks] Find the coefficients ψ_j, $j = 0, 1, 2, \ldots$, in the representation

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

of the ARMA$(2, 1)$ process,

$$(1 - 0.5B + 0.04B^2)X_t = (1 + 0.25B)Z_t, \qquad \{Z_t\} \sim \text{WN}(0, \sigma^2).$$

3. [5 marks] Let $\{X_t\}$ be the process defined by

$$X_t = A\cos\left(\frac{\pi}{3}t\right) + B\sin\left(\frac{\pi}{3}t\right) + Y_t$$

where $Y_t = Z_t + 2.5Z_{t-1}$, $\{Z_t\} \sim \text{WN}(0, \sigma^2)$, A and B are uncorrelated with mean 0 and variance v^2, and Z_t is uncorrelated with A and B for each t. Find the **autocovariance function** and **spectral distribution** function of $\{X_t\}$.

4. [5 marks] Determine the autocovariance function of the process with spectral density

$$f(\lambda) = \frac{\pi - |\lambda|}{\pi^2}, \qquad -\pi \leq \lambda \leq \pi.$$

5. [10 marks] Let X_1, X_2, X_4, X_5 be observations from the MA(1) model

$$X_t = Z_t + \theta Z_{t-1}, \qquad \{Z_t\} \sim \text{WN}(0, \sigma^2).$$

a) Find the best linear estimate of the missing value X_3 in terms of X_1 and X_2.

b) Find the best linear estimate of the missing value X_3 in terms of X_4 and X_5.

c) Find the best linear estimate of the missing value X_3 in terms of X_1, X_2, X_4 and X_5.

d) Compute the mean-squared errors for each of the estimates in (a), (b), and (c).

(for graduate students only)

6. [10 marks] The spectral density of a process $\{X_t\}$ is defined on $[0, \pi]$ by

$$f(\lambda) = \begin{cases} 100, & \frac{\pi}{6} - 0.01 < \lambda < \frac{\pi}{6} + 0.01. \\ 0, & \text{otherwise} \end{cases}$$

and on $[-\pi, 0]$ by $f(\lambda) = f(-\lambda)$.

(a) Evaluate the covariance function of $\{X_t\}$ at lags 0 and 1.

(b) What is the variance of $Y_t = X_t - X_{t-12}$?

7. [10 marks] Implement one of the following two routines in R/MATLAB. Hand in clearly commented code **with test cases** to show proof-of-completion.

(a) Implement a Slepian taper-based multitaper spectrum estimate. Do non-adaptive weighting, and allow for zeropadding. Feel free to steal the code to generate the DPSS tapers, either from `multitaper` (in R) or any other source you wish.

(b) Implement an 'innovations' creator, that, given a series of data points, will return the innovations corresponding to them. That is, given $X_1, X_2, \ldots, X_n$, return $X_1 - \widehat{X}_1, X_2 - \widehat{X}_2, \ldots, X_n - \widehat{X}_n$. The method you use is entirely up to you, although you should reference formulas from the text(s) that you are using in your code.

8. [5 marks] Given two observations x_1 and x_2 from the causal AR(1) process satisfying

$$X_t = \phi X_{t-1} + Z_t, \qquad \{Z_t\} \sim \mathrm{WN}(0, \sigma_Z^2)$$

and assuming that $|x_1| \neq |x_2|$, find the maximum likelihood estimates of ϕ and σ_Z^2.