1. [8 marks] (R problem) Generate a realization of the AR(2) process

$$X_t = 0.75X_{t-1} - 0.5X_{t-2} + Z_t, \qquad \{Z_t\} \sim WN(0, \sigma^2)$$

of length 500 and using spectrum(), estimate the periodogram of the series. Using the equation from your text (and class), determine the true spectral density and overplot it **on the same plot** as the spectrum using a different colour. Be sure to hand in your code for doing all of this, as well as a clean plot.

2. [10 marks] Given  $\{Z_t\}$  an orthogonal increment process, and

$$D_N(f) = \frac{\sin(N\pi f)}{N\sin(\pi f)}$$

the Dirichlet kernel, with given relationship

$$\sum_{t=1}^{N} e^{i2\pi ft} = \begin{cases} N e^{i(N+1)\pi f} D_N(f) & f \neq 0, \pm 1, \pm 2, \dots \\ N & f = 0, \pm 1, \pm 2, \dots \end{cases}$$

show that

$$\frac{1}{\sqrt{N}} \sum_{t=1}^{N} X_t e^{-i2\pi ft} = \sqrt{N} \int_{-1/2}^{1/2} e^{i(N+1)\pi(f'-f)} D_N(f'-f) dZ(f')$$

with  $|f| \leq 1/2$ . You should use the (process) spectral representation theorem.

3. [6 marks] Show that

$$\int_{-\pi}^{\pi} e^{i(h-k)\lambda} d\lambda = \begin{cases} 2\pi & \text{, if } k = h \\ 0 & \text{, else} \end{cases}$$

(this is Question 4.1 in your text).

4. [16 marks] Suppose that  $\{X_t\}$  is the noncausal and noninvertible ARMA(1,1) process satisfying

$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}, \qquad \{Z_t\} \sim WN(0, \sigma^2),$$

where  $|\phi| > 1$  and  $|\theta| > 1$ . Define  $\tilde{\phi}(B) = 1 - \frac{1}{\phi}B$  and  $\tilde{\theta}(B) = 1 + \frac{1}{\theta}B$ , and let  $\{W_t\}$  be the process given by

$$W_t := \tilde{\theta}^{-1}(B)\tilde{\phi}(B)X_t.$$

- a) Show that  $\{W_t\}$  has a constant spectral density function.
- b) Conclude that  $\{W_t\} \sim WN(0, \sigma_W^2)$ . Give an explicit formula for  $\sigma_W^2$  in terms of  $\phi, \theta$ , and  $\sigma^2$ .
- c) Deduce that  $\tilde{\phi}(B)X_t = \tilde{\theta}(B)W_t$ , so that  $\{X_t\}$  is a causal and invertible ARMA(1,1) process relative to the white noise sequence  $\{W_t\}$ .

(for graduate students, or undergraduates seeking extra credit)

5. [20 marks] Using mvfft() in R (or equivalent in MATLAB), implement a **direct spectral estimator**. Your routine should have support for zero-padding and tapering, and should also generate a suitable plot, modifiable by logical parameter passing. For the plotting, you should follow the pattern set by your environment: in R, this means presenting a plot similar in form to the spectrum() routine, which actually has a default spec object and associated plotting routine. Feel free to fall through to this routine.

You should test your routine against the setup from Question 1, presenting an example of both a periodogram estimate (i.e. taper=  $1/\sqrt{N}$ ), and a tapered spectral estimator. The taper example can be chosen at your discretion.