1. [6 marks] (R problem) Using the built-in sunspot data file in R (`data(sunspots)`), find the Yule-Walker estimates of ϕ_1, ϕ_2 and σ^2 in the model

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + Z_t, \qquad \{Z_t\} \sim \text{WN}(0, \sigma^2).$$

You should use the built-in (`stats` package) `ar()` function. The `method=` parameter lets you set which method the routine will use. Also fit this model using ordinary least-squares and maximum-likelihood estimation, and compare the three fits. Which fit is the best?

2. [8 marks] Consider the AR(2) process $\{X_t\}$ satisfying

$$X_t - \phi X_{t-1} - \phi^2 X_{t-2} = Z_t, \qquad \{Z_t\} \sim \text{WN}(0, \sigma^2).$$

a. For which values of ϕ is this a causal process?

b. The following sample moments were computed after observing $X_1, \ldots, X_{200}$:

$$\widehat{\gamma}_0 = 6.06 \qquad \widehat{\rho}_1 = 0.687.$$

Find estimates of ϕ and σ^2 using the Yule-Walker equations; if you find more than one solution, choose the one that is causal.

(this is Question 5.3 in your text)

3. [8 marks] The sunspot numbers $\{X_t, t = 1, 2, \ldots, 100\}$ used in the text (not precisely the same as the set used in Problem 1 above) have sample autocovariances $\widehat{\gamma}_0 = 1382.2$, $\widehat{\gamma}_1 = 1114.4$, $\widehat{\gamma}_2 = 591.73$ and $\widehat{\gamma}_3 = 96.216$. Using these values, find the Yule-Walker estimates of ϕ_1, ϕ_2, and σ^2 in the model

$$Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + Z_t, \qquad \{Z_t\} \sim \text{WN}(0, \sigma^2),$$

for the mean-corrected series $Y_t = X_t - 46.93$, $t = 1, \ldots, 100$. Assuming that the data really are a realization of an AR(2) pocess, find 95% confidence intervals for ϕ_1 and ϕ_2.

4. [8 marks] Show that the value at lag 2 of the PACF of the MA(1) process

$$X_t = Z_t + \theta Z_{t-1}, \qquad t = 0, \pm 1, \ldots,$$

where $\{Z_t\} \sim \text{WN}(0, \sigma^2)$ is

$$\alpha(2) = \phi_{22} = \frac{-\theta^2}{1 + \theta^2 + \theta^4}.$$

You should use the best linear predictor form to set up your linear equations, then solve.
(this is Question 3.11 in your text)

5. [10 marks] The values $0.644, -0.422, -0.919, -1.573, 0.852, -0.907, 0.686, -0.753, -0.954, 0.576$ are simulated values of $X_1, \ldots, X_{10}$ where $\{X_t\}$ is the ARMA(2,1) process,

$$X_t - 0.1X_{t-1} - 0.12X_{t-2} = Z_t - 0.7Z_{t-1}, \qquad \{Z_t\} \sim \text{WN}(0, 1).$$

a. Compute the forecasts $\mathbf{P}_{10}X_{11}$ and $\mathbf{P}_{10}X_{12}$, and the corresponding mean-squared error, using the innovations algorithm.

b. Assuming that $\{Z_t\} \sim \mathcal{N}(0, 1)$, construct 95% prediction bounds for X_{11} and X_{12}.

(the actual simulated values were $X_{11} = .074$ and $X_{12} = 1.097$)

(for graduate students, or undergraduates seeking extra credit)

6. [20 marks] Implement the recursive innovations algorithm in the programming language of your choice (suggested language is R, to save heavy lifting). Set up your routine to perform 1-step linear prediction $\widehat{X}_{n+1}$, as per Equation (2.5.28) in your text. This will require that you take the set of $\{X_t\}$'s as input. Allow for the user to pass in the ACVF, or alternatively, to have your routine compute the sample ACVF using `acf()` (in R; your choice of alternative in MATLAB).

You should test your routine by simulating samples of the ARMA(2,1) process given in Problem 5. Estimate several sets of samples, and pass your routine $n - 1$ of them to evaluate performance on the prediction of $\widehat{X}_n$.