1. [6 marks] (R problem) Using `arima.sim`, generate examples of MA(2), AR(2), and (using the same coefficients) ARMA(2,2) time series. Plot the ACF and PACF for each of these, using the `par(mfrow=c(3,1))` command to fit 3 plots on a single page, putting the three ACFs together and the three PACFs together. Write a short discussion of the features you see in these plots.

2. [7 marks] Consider two stationary time series

$$X_t = 0.4X_{t-1} + \beta_t$$
$$Y_t = Z_t - 3Z_{t-1}$$

where $\beta_t \sim \text{WN}(0, \sigma_\beta^2)$ and $Z_t \sim \text{WN}(0, \sigma_Z^2)$ are independent white noise processes, i.e $\mathbf{E}\left[\beta_t Z_s\right] = 0$ for all s, t.

a) Find the ACVF of $H_t = X_t + Y_t$, and verify that H_t is stationary.

b) Determine which ARMA(p,q) model best fits H_t.

3. [5 marks] Compute the first four lags of the partial autocorrelation function for an arbitrary $\text{MA}(2)$ process, similar to how we computed the PACF for an arbitrary $\text{AR}(2)$ process in class.

4. [8 marks] Let $\{X_t\}$ denote the unique stationary solution of the autoregressive equations

$$X_t = \phi X_{t-1} + Z_t, \qquad t = 0, \pm 1, \ldots$$

where $\{Z_t\} \sim \text{WN}(0, \sigma_Z^2)$ and $|\phi| > 1$. Then X_t can be expressed as

$$X_t = -\sum_{j=1}^{\infty} \phi^{-j} Z_{t+j}.$$

Define the new sequence

$$W_t = X_t - \frac{1}{\phi} X_{t-1},$$

show that $\{W_t\} \sim \text{WN}(0, \sigma_W^2)$, and express σ_W^2 in terms of σ_Z^2 and ϕ. These calculations show that $\{X_t\}$ is the (unique stationary) solution of the *causal* AR equations

$$X_t = \frac{1}{\phi} X_{t-1} + W_t, \qquad t = 0, \pm 1, \ldots.$$

(this is Question 3.8 in the text)

5. [12 marks] Read in the data set provided (`Assign3_ARMA.dat`) on the website, and using everything you know about ACFs and PACFs, determine a suitable model to represent this data. Once you have the model, compute the theoretical ACVF and plot the sample ACVF on the same set of axes as the theoretical ACVF, and discuss any differences you see. **Note: you should use the arma() function in R to find the model** (from the `tseries` package). You are **not** expected to use the current material on model-fitting algorithms to do this work manually, but are expected to determine the model **order** as best you can.

(for graduate students, or undergraduates seeking extra credit)

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6. [10 marks] Using the MA() sample generator that you created for Assignment 2, generate $m = 100$ different realizations of a MA(3) for sample lengths $n = \{50, 100, 200, 500\}$. To be clear, fix some $\theta = \{\theta_0, \theta_1, \theta_2, \theta_3\}$, then for each n in the set of lengths, generate $m = 100$ different realizations. In practice, this will be accomplished by calling your routine from Assignment 2 $m = 100$ times with the same n and θ.

For each set of $m = 100$ realizations, use the `arma()` function from the `tseries` package (or equivalent in your programming language of choice) to fit an MA(3) model to each realization of sample data. Generate one plot per parameter showing the bias in the estimate as a function of n (i.e. y is bias, x is n). Discuss these results, especially in the context of the *sample autocorrelation* that is at the heart of the estimator. You should review Section 2.4 in your text.

Note that **bias** of an estimator is defined as the difference between this estimator's expected value and the true value of the parameter being estimated. In this case, you know the true value (since you generated the MA(q) in the first place), and the expected value will be the average of the 100 different realizations for each n.

(Aside: being able to fix $q = 3$ in the calls to `arma()` is cheating. Think about how we might choose q in general.)

7. [5 marks] Generalize your work from the previous question and create a function that takes q, θ (as a vector), n (as a vector), and m, and returns the data necessary to create the plots that you displayed. Test this routine by passing it the inputs from the previous question and ensure the outputs are the same.

8. [5 marks] For a MA(2) process, find the largest possible values of $|\rho(1)|$ and $|\rho(2)|$.