1. [6 marks] (R problem) Generate 100 samples from a MA(2) process using the `arima.sim()` function, make a time series plot of the data, then calculate the sample autocorrelation function and sample partial autocorrelation function (PACF). Plot the correlogram and the sample PACF.

2. [7 marks] Let $\{Y_t\}$ be the AR(1) plus noise model given by

$$Y_t = X_t + \zeta_t$$

where $\{\zeta_t\} \sim \text{WN}(0, \sigma_\zeta^2)$ and $\{X_t\}$ is the AR(1) process with correlation ϕ and variance σ_x^2, i.e.

$$X_t - \phi X_{t-1} = Z_t,\ \{Z_t\} \sim \text{WN}(0, \sigma_z^2),$$

and $\mathbb{E}[\zeta_s Z_t] = 0$ for all s and t.

i) Show that $\{Y_t\}$ is stationary and find its autocovariance function.

ii) Show that the time series $U_t := Y_t - \phi Y_{t-1}$ is $1-$correlated. It follows that U_t is an MA(1) process.

iii) Conclude that $\{Y_t\}$ is an ARMA(1,1) process and express the three parameters of this model in terms of ϕ, σ_ζ^2 and σ_x^2.

(this is question 2.9 in your text)

3. [5 marks] Question 3.1 in your text.

4. [6 marks] Show that the two MA(1) processes

$$\begin{aligned} X_t &= Z_t + \theta Z_{t-1}, & \{Z_t\} &\sim \text{WN}(0, \sigma^2) \\ Y_t &= \tilde{Z}_t + \theta^{-1} \tilde{Z}_{t-1}, & \{\tilde{Z}_t\} &\sim \text{WN}(0, \sigma^2\theta^2), \end{aligned}$$

where $0 < |\theta| < 1$, have the same autocovariance functions.
(this is question 3.6 in your text)

5. [6 marks] Consider the ARMA(2,1) model

$$X_t = 1.8X_{t-1} - 0.81X_{t-2} + Z_t + 0.5Z_{t-1}.$$

i) Is this model causal?

ii) Is this model invertible?

iii) Compute the first four coefficients of the causal representation

$$X_t = \sum_{j=0}^{\infty} \beta_j z_{t-j}.$$

(for graduate students, or undergraduates seeking extra credit)

6. [18 marks] Implement a function that generates n realizations for a MA(q) process. That is, your function should generate a suitable number of random variables $\{Z_t\} \sim \text{WN}(0, \sigma^2)$ (where σ^2 is

a parameter), and, taking $\theta_1, \ldots, \theta_q$ as additional parameters, return $X_t = Z_t + \theta_1 Z_{t-1} + \cdots + \theta_q Z_{t-q}$. Ensure your function is sufficiently generalized so as to allow for a range of n, q, and σ^2 parameters to be set, and ensure that you include a check on the values of the θ_i's.

You should feel free to do some research to find the best (or at least a reasonable) method for simulating the $\{Z_t\}$ values. **Do not implement your own random number generator.**

7. [6 marks] For the ARMA(1,1) process given by

$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1},$$

(where $\{Z_t\} \sim \text{WN}(0, \sigma^2)$ and $\phi + \theta \neq 0$), show that the autocorrelation function for lag $h \geq 2$ is

$$\rho(h) = \phi^{h-1} \rho(1).$$

8. [6 marks] Show that in order for an AR(2) process with autoregressive polynomial $\phi(z) = 1 - \phi_1 z - \phi_2 z^2$ to be causal, the parameters (ϕ_1, ϕ_2) must lie in the triangular region determined by the intersection of the three regions,

$$\begin{aligned} \phi_2 + \phi_1 &< 1, \\ \phi_2 - \phi_1 &< 1, \\ |\phi_2| &< 1. \end{aligned}$$