QUEEN'S UNIVERSITY
APSC 171J – Quiz #1: Solutions
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INSTRUCTIONS

- This quiz is being written in the tutorial (9:30-10:20am) Wednesday, January 30
- Answer all questions, writing clearly on the sheets provided.
- One mark in each question is for a **fully** correct solution, which **must** be placed in the box provided
- Whenever possible, simplify your solution.
- There are no part marks: you will receive only integer marks for each question.
- The quiz is **double sided** – make sure you look at both sides of each sheet of paper!

FOR INSTRUCTOR'S USE ONLY		
Question	Mark Available	Received
1	4	
2	4	
3	8	
TOTAL	16	

1. [4 marks] Find $\frac{dy}{dx}$ if

$$y = \left(3x^2\right)^{\cos^2(x)}.$$

([**3 marks**] for process, [**1 mark**] for final answer in box below)

There are two techniques we learned for solving one of these problems:

1. The first is to treat f^g as both f constant and g constant, in a product-rule-like expansion. If

$$f(x) = 3x^2 \qquad \text{and} \qquad g(x) = \cos^2(x) \qquad \text{with } y(x) = (f(x))^{g(x)}$$

then $y' = g \cdot f^{g-1} \cdot f' + f^g \cdot g' \cdot \ln(f)$.

$$\begin{aligned}
\frac{d}{dx}(y) &= \cos^2(x)\left(3x^2\right)^{\cos^2(x)-1} \cdot (6x) + \left(3x^2\right)^{\cos^2(x)} \cdot (-2\cos(x)\sin(x)) \cdot \ln(3x^2) \\
&= 2\cos(x)\left(3x^2\right)^{\cos^2(x)}\left[\frac{6x\cos(x)}{6x^2} - \sin(x)\ln(3x^2)\right] \\
&= 2\cos(x)\left(3x^2\right)^{\cos^2(x)}\left[\frac{\cos(x)}{x} - \sin(x)\ln(3x^2)\right]
\end{aligned}$$

2. The second is to take the natural log of both sides:

$$\ln(y(x)) = \ln\left[(3x^2)^{\cos^2(x)}\right] = \cos^2(x)\ln\left[3x^2\right]$$

and then differentiate implicitly

$$\frac{1}{y}y' = 2\cos(x)\left[-\sin(x)\right]\ln\left[3x^2\right] + \cos^2(x)\frac{1}{3x^2} \cdot (6x)$$

which simplifies to

$$\begin{aligned}
y' &= y\left[\cos^2(x)\frac{2}{x} - 2\sin(x)\cos(x)\ln\left(3x^2\right)\right] \\
&= (3x^2)^{\cos^2(x)}\left[2\cos(x)\left(\frac{\cos(x)}{x} - \sin(x)\ln\left(3x^2\right)\right)\right] \\
&= 2\cos(x)\left(3x^2\right)^{\cos^2(x)}\left[\frac{\cos(x)}{x} - \sin(x)\ln(3x^2)\right]
\end{aligned}$$

Final Answer:

$$\boxed{y'(x) = 2\cos(x)\left(3x^2\right)^{\cos^2(x)}\left[\frac{\cos(x)}{x} - \sin(x)\ln(3x^2)\right]}$$

2. [4 marks] Find $\frac{dy}{dx}$ if

$$\tan(xy) = \frac{x}{y}.$$

(**[3 marks]** for process, **[1 mark]** for final answer in box below)

Recall that $\frac{d}{dx}(\tan(x)) = \sec^2(x)$. This is the easiest way to do the problem. You could also remember that $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and then use the quotient rule, but this is **very** messy.

$$\begin{aligned}
\frac{d}{dx}(\tan(xy)) &= \frac{d}{dx}\left(\frac{x}{y}\right) \\
\sec^2(xy)\cdot\frac{d}{dx}(xy) &= \frac{(1)y - xy'}{y^2} \qquad \text{(Chain and Quotient, Implicitly)} \\
\sec^2(xy)\cdot(y + xy') &= \frac{y - xy'}{y^2} \\
y^2\left[\sec^2(xy)(y + xy')\right] &= y - xy' \\
y^3\sec^2(xy) + xy^2y'\sec^2(xy) + xy' &= y \\
xy^2y'\sec^2(xy) + xy' &= y - y^3\sec^2(xy) \\
y'\left[xy^2\sec^2(xy) + x\right] &= y - y^3\sec^2(xy) \\
y'(x) &= \frac{y - y^3\sec^2(xy)}{xy^2\sec^2(xy) + x}.
\end{aligned}$$

Final Answer:

$$\boxed{y'(x) = \frac{y - y^3\sec^2(xy)}{xy^2\sec^2(xy) + x}}$$

3. [8 marks] Sheets of aluminum are used to construct cylindrical drinking cans: one sheet is used for the sides, and one disc for each of the top and the bottom of the cans. The sides are made from aluminum sheets with density 0.054 g/cm^2, while the top and bottom are made from thicker aluminum with density 0.090 g/cm^2. Find the height and radius of a 300 mL cylindrical can that uses the minimum **mass** of aluminum. Make sure you include an explanation why the dimensions you found really give a **global** minimum mass. *Note: mass is density times volume, but in the formulation given here, mass is density times surface area, and we assume the thickness of the aluminum is factored into the density value.*

(**[2 marks]** for optimizing equation, **[3 marks]** for process, **[1 mark]** for classification as minimum, **[1 mark]** for demonstration of global property and **[1 mark]** for final answer.)

Please use the back of **this** sheet of paper for extra space and your final solution.

The trickiest part of this question was writing the proper optimizing equation. Recall what we know about cylinders:

$$V = \pi r^2 h \qquad \text{and } S_{area} = 2\pi r^2 + 2\pi r h$$

We are given $V = 300\text{cm}^3 = 300$ mL, so this will be our constraint. Now, we are asked to minimize the **mass** of aluminum used, so we have to find a mass equation. Recall that mass is volume times density, but in this case, is given as surface area times density. Thus, apply the density for the top/bottom to that portion of the surface area equation, and the other density to the sides portion:

$$m(r,h) = 2\pi r^2 \cdot 0.090 + 2\pi r h \cdot 0.054$$

Now (if this next step is not familiar to you, refer to the algorithm for solving optimization problems that I posted to the website), we substitute the constraint into this. You **could** solve the constraint for r, but that is very silly, as it introduces a square root that we would rather avoid. Instead, write

$$300 = \pi r^2 h \qquad \Rightarrow \qquad h = \frac{300}{\pi r^2}$$

and upon substitution

$$m(r,h) = 2\pi r^2 \cdot 0.090 + 2\pi r \left(\frac{300}{\pi r^2}\right) \cdot 0.054$$

Combining all our constants together to make life simpler, we have

$$m(r) = 0.5655 r^2 + \frac{32.4}{r}.$$

This equation determines the mass as a function of the radius r only, and is our optimizing equation. Just getting this equation was worth 3 marks. Now, differentiate this equation with respect to r and then set it equal to zero and solve for the critical points:

$$\begin{aligned}
m'(r) &= 1.13097r - \frac{32.4}{r^2} = 0 \\
1.13097r^3 &= 32.4 \\
r^3 &= 28.64789 \\
r &= 3.0598 \approx 3.06\text{cm}.
\end{aligned}$$

Now, substitute this back into the constraint equation to get h:

$$h = \frac{300}{\pi(3.06^2)} = 10.1994 \approx 10.2\text{cm}.$$

If you got this far, you earned 6 marks. The final two marks were for showing that the point $r = 3.06$ was a relative minimum, and justifying why it is, in fact, an absolute / global minimum.

To confirm local minimum, we use the first derivative test:

r	$r < 3.06$	$r = 3.06$	$r > 3.06$
m′(r)	$-$	0	$+$

where we use $r = 3.0$ and $r = 3.1$ as our "check" points. Thus, by the First Derivative Test, the critical point $r = 3.06$ is a relative minimum.

Since there is only one critical point on the domain, and since the slopes of the function are negative to the left of the critical point, we know that the slopes remain negative as $r \to 0$. Similarly, since the slopes of the function to the *right* of the critical point are positive, they also cannot change, and the slopes remain positive as $r \to \infty$. This is sufficient for us to claim that the critical point found, $r = 3.06$, is a global minimum point.

Thus, the minimum mass of aluminum used will occur for a can of dimensions $r = 3.06$cm and height $h = 10.2$cm.

Final Answer: $r = 3.06$cm and $h = 10.2$cm