APSC 171
2011 December Exam
Solutions

Question 1:

To compute the velocity equation, we have to differentiate the third term of this vector: $[\ln(t)]^t$. This is another example (which appears to show up every year on the exam) of a $f(x)^{g(x)}$ derivative. Rewrite it as:

$$[\ln(t)]^t = e^{\ln\left[[\ln(t)]^t\right]} = e^{t\ln[\ln(t)]}.$$

Then, differentiate:

$$\begin{aligned} e^{t\ln[\ln(t)]} \cdot \frac{d}{dt}\left(t\ln[\ln(t)]\right) &= e^{t\ln[\ln(t)]} \cdot \left(\ln[\ln(t)] + t\frac{1}{\ln(t)}\frac{1}{t}\right) \\ &= [\ln(t)]^t \cdot \left(\ln[\ln(t)] + t\frac{1}{\ln(t)}\frac{1}{t}\right) \\ &= [\ln(t)]^{t-1} + [\ln(t)]^t\ln[\ln(t)]. \end{aligned}$$

Thus, the velocity vector is:

$$\vec{s}'(t) = \langle \cos(\tan(t))\sec^2(t), 3t^2\cdot 2^t + t^2\cdot 2^t\ln(2), [\ln(t)]^{t-1} + [\ln(t)]^t\ln[\ln(t)]\rangle.$$

Question 2:

Use Integration by Substitution, letting $u = x^{2/3}$ which gives $du/dx = \frac{2}{3}x^{-1/3}$ which gives $dx = dx = \frac{3}{2}dux^{1/3}$. Then:

$$\begin{aligned} \int_1^8 x^{-1/3}\sin(\pi x^{2/3})dx &= \int_1^8 x^{-1/3}\sin(\pi u)\frac{3}{2}dux^{1/3} \\ &= \int_1^8 \frac{3}{2}\sin(\pi u)du \\ &= -\frac{3}{2\pi}\cos(\pi u)\Big|_{x=1}^{x=8} \\ &= -\frac{3}{2\pi}\cos(\pi x^{2/3})\Big|_{x=1}^{x=8} \\ &= -\frac{3}{2\pi}\cos(4\pi) + \frac{3}{2\pi}\cos(\pi) \\ &= -\frac{3}{\pi}. \end{aligned}$$

Question 3:

Implicitly differentiate:

$$2xy^2 + 2x^2yy' + \frac{1}{1+x^2}y + \arctan(x)y' = 0$$

and then rearrange

$$y' = \frac{-2xy^2 - \frac{y}{1+x^2}}{2x^2y + \arctan(x)}.$$

Question 4:

To solve this integral, use Partial Fractions:

$$\frac{x-1}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2} \qquad \Rightarrow A(x+1) + B = x - 1$$

so $A = 1$ and $B = -2$. Then:

$$\begin{aligned}
\lim_{T\to\infty} \int_0^T \left(\frac{1}{x+1} - \frac{2}{(x+1)^2} \right) dx &= \lim_{T\to\infty} \left[\ln|x+1| + \frac{2}{x+1} \right]_{x=0}^{x=T} \\
&= \lim_{T\to\infty} \left[\ln|T+1| + \frac{2}{T+1} - \ln|1| - 2 \right] \\
&= \infty
\end{aligned}$$

so the integral **diverges**.

Question 5:

(a) Recall that speed is the magnitude of velocity, so find the velocity vector:

$$v(t) = \vec{s}'(t) = \langle \frac{1}{2}t^{-3/2}, 2t \rangle$$

and then take its magnitude

$$\text{speed } = \sqrt{\frac{1}{4}t^{-3} + 4t^2}$$

and then evaluating at $t = 1$, get speed$(1) = \sqrt{17}/2$.

(b) To eliminate the variable t, start by writing

$$x = -t^{-1/2} \qquad y = t^2 + 1$$

and then solve the first for t:

$$t = (-1/x)^2 \qquad \text{with } x < 0 \text{ for } t \geq 0.$$

Then substitute this t into the second equation:

$$y = (-1/x)^4 + 1 = \frac{1}{x^4} + 1 \qquad x < 0.$$

This curve starts at $(-\infty, 1)$ and goes to $(0, \infty)$ as $t \to \infty$. This corresponds to the half-function $y = x^{-4} + 1$ for $x < 0$.

(c) Given the two points $x = 1, y = 2$ and $x = 0, y = \infty$, we eliminate the second curve. Then, because $y = x^{-4}+1$, $y > 0$ for all x, so it must be curve 1. At $t = 1$, $x = 1, y = 2$ and as $t \to \infty$ x goes to 0 and y goes to ∞.

(d) To determine whether the particle's speed is increasing or decreasing, compute the acceleration vector:

$$\vec{a}(t) = \langle -3/4(t)^{-5/2}, 2\rangle$$

so that $\vec{a}(1) = \langle -3/4, 2\rangle$ so $|\vec{a}(1)| = \sqrt{9/16+4} > 0$ so the particle is speeding up at time $t = 1$.

Question 6:

(a) The question gives us that $l' = 3$, $V' = 120$ and $w' = h'$. Write the volume equation for the sponge:

$$V = lwh$$

then implicitly differentiate it

$$V' = l'wh + lw'h + lwh'$$

and then evaluate

$$120 = 3(5)(5) + 20(w')(5) + 20(5)(h')$$

so $120 = 75 + 150w'$ and $w' = h' = 45/150 = 0.3$ cm/min.

(b) Begin with the surface area equation:

$$SA = 2lw + 2lh + 2wh$$

then implicitly differentiate it:

$$(SA)' = 2l'w + 2lw' + 2l'h + 2lh' + 2w'h + 2wh'$$

and substitute the known values:

$$(SA)' = 2\left[3(5) + 5(0.3) + 3(5) + 5(0.3) + 0.3(5) + (0.3)(5)\right] = 72cm^2/min.$$

Question 7:

The shape in question between $y = 1 - x^2$ and $y = 0$ is a concave down parabola with a flat bottom. It runs from $x = -1$ to $x = 1$, with maximum at $x = 0$. We are rotating about $x = 1$, so cylindrical shells make the most sense. Their radii will run from $r = 0$ to $r = 2$ and correspond to $1 - x$. The height at any point x will be $y = 1 - x^2$. Thus, we can find the cylindrical shell volume equation:

$$V = 2\pi r h \Delta r = 2\pi(1 - x)(1 - x^2)\Delta x$$

and then set up the integral and solve:

$$\begin{aligned} V &= \int_{-1}^{1} 2\pi(1 - x)(1 - x^2)dx \\ &= 2\pi \left[x - \frac{x^3}{3} - \frac{x^2}{2} + \frac{x^4}{4} \right]_{x=-1}^{x=1} \\ &= 2\pi \, [5/12] = \frac{5}{6}\pi. \end{aligned}$$

Question 8:

On the cross-sectional view, draw a horizontal line across the box. Then draw a vertical line from the bottom right corner straight up, perpendicularly, so that it intersects the top of the box at the point 10 cm from the top left. This is a triangle, height 15 and width 3. Then, if you consider an arbitrary height h from the bottom of the box, the cross-sectional width will be 10 plus a similar triangle piece that comes from

$$\frac{h}{15} = \frac{w}{3}$$

so $w = h/5$. Then the cross-sectional width will be $10 + h/5$. The slice volume will be

$$75 \cdot (10 + h/5)\Delta h.$$

Multiply this volume by $1000kg/m^3$, and **convert to meters**!

$$m = 0.75(0.1 + h/5) \cdot 1000kg.$$

Now, remember that gravitational potential energy is the same as the work required to lift the water to that height, so that:

$$E = W = \int_{h=0}^{h=0.15} (0.75(0.1 + h/5) \cdot 1000) \cdot 9.8 \cdot (1.5 + h)dh$$
$$= 0.75(9.8)(1000) \int_0^{0.15} (0.1 + h/5)(1.5 + h)dh$$
$$= 0.75(9.8)(1000) \int_0^{0.15} \left[0.15 + 0.1h + 0.3h + 0.2h^2\right] dh$$
$$= 0.75(9.8)(1000) \left[0.15h + 0.05h^2 + 0.15h^2 + 1/15h^3\right]_{h=0}^{h=0.15}$$
$$= 200.1J.$$

Question 9:

(a) See the Figure:

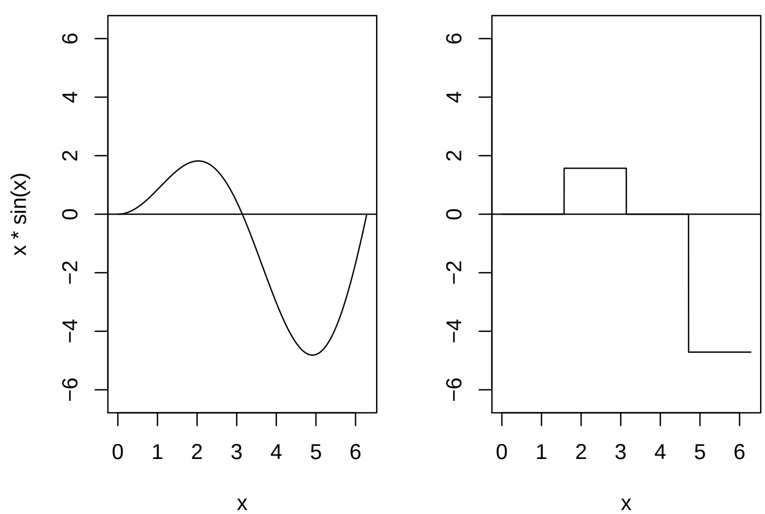

Figure 1: Figure for Q9 (a).

(b) See Figure.

(c) To compute the trapezoidal rule:

$$\text{TRAP}_4(f) = \frac{\pi/2}{2}\left[2\frac{\pi}{2} - \frac{3\pi}{2}\right] = \pi^2/4 - 3\pi^2/8 \approx -1.23.$$

(d) Now, use Integration by Parts:

$$\begin{array}{ll} u = x & v = -\cos(x) \\ du = dx & dv = \sin(x)dx \end{array}$$

so that

$$\int_0^{2\pi} x\sin(x)dx = -x\cos(x)\Big|_{x=0}^{x=2\pi} + \int_0^{2\pi}\cos(x)dx = -2\pi + \sin(x)\Big|_0^{2\pi} = -2\pi \approx -6.28.$$

Question 10:

(a) Recall the general form of the solution ($\sin + \cos$). Then write

$$x(t) = A\sin(\omega t) = \sin\left(\sqrt{16/25}x\right).$$

This is a solution.

(b) Now, start with the full general solution:

$$x(t) = A\sin(4/5x) + B\cos(4/5x).$$

Now, use the $x(0)$ initial condition:

$$x(0) = 3 = A(0) + B(1)$$

which gives $B = 3$. Then, $x'(0) = 4$ so

$$x'(0) = 4 = 4/5A\cos(4/5\cdot 0) - 4/5B\sin(4/5\cdot 0)$$

so $A = 5$. Thus:

$$x(t) = 5\sin(4/5x) + 3\cos(4/5x).$$

(c) The period of oscillations is $2\pi/\omega = 2\pi/(4/5) = 2.5\pi$.

(d) If the mass were to be doubled, then $\omega = 4/5$ goes to $\omega_2 = \sqrt{16/50} = 4/5\sqrt{2}$. Thus, the period will go from 2.5π to $\frac{5\sqrt{2}}{2}\pi$ which is increased by a multiplicative factor of $\sqrt{2}$.

Question 11:

(a) The differential equation is simple enough:

$$\frac{dH}{dt} = 4 - 0.8H.$$

(b) To solve this, recognize that it is a separable differential equation, so

$$\begin{aligned}
\int \frac{dH}{4-0.8H} &= \int dt \\
\frac{1}{-0.8}\ln|4-0.8H| &= t + C \\
4 - 0.8H &= Ce^{-0.8t} \\
H(t) &= \frac{4 - Ce^{-0.8t}}{0.8} \\
H(t) &= 5 - Ce^{-0.8t} \\
&\text{so apply the initial condition} \\
H(0) &= 0 = 5 - Ce^{0} \qquad C = 5 \\
H(t) &= 5 - 5e^{-0.8t}.
\end{aligned}$$

(c) Compute the limit as $t \to \infty$:

$$\lim_{t\to\infty} H(t) = 5 - \lim_{t\to\infty} 5e^{-0.8t} = 5.$$

so the mass of heparin converges to 5mg over time, with constant input.

(d) So $H(0) = 12$mg, and the cut-off is 0.5mg. There is no input, so the differential equation is different:

$$\frac{dH}{dt} = -0.8H$$

which has solution

$$\begin{aligned}
\int \frac{dH}{H} &= \int (-0.8)dt \\
\ln|H(t)| &= -0.8t + C \\
H(t) &= Ce^{-0.8t} \\
&\text{apply the IC} \\
H(0) &= 12 = Ce^{0} \qquad \Rightarrow C = 12 \\
H(t) &= 12e^{-0.8t}.
\end{aligned}$$

Now, set

$$0.5 = 12e^{-0.8t} \qquad \Rightarrow \qquad \ln|0.5/12| = -0.8t$$

so that $t = 3.97 \approx 4$ hours.