APSC 171
2010 December Exam
Solutions

Question 1:

This is a standard "exponential trick" product rule problem. Since we have $y = (f(x))^{g(x)}$, we have to rewrite the function as:

$$y = e^{\ln\left((\arctan(3x))^{x+1}\right)}$$

and then remember that $\ln(f(x)^{g(x)})$ can be rewritten as $g(x)\ln(f(x))$. Thus:

$$y = e^{(x+1)\ln(\arctan(3x))}.$$

Now, differentiate:

$$\begin{aligned}
\frac{dy}{dx} &= e^{(x+1)\ln(\arctan(3x))} \cdot \frac{d}{dx}\left[(x+1)\ln(\arctan(3x))\right] \\
&= e^{(x+1)\ln(\arctan(3x))}\left[\ln(\arctan(3x)) + (x+1)\frac{1}{\arctan(3x)}\frac{1}{1+(3x)^2}\cdot 3\right] \\
&= [\arctan(3x)]^{x+1}\left[\ln(\arctan(3x)) + (x+1)\frac{1}{\arctan(3x)}\frac{1}{1+(3x)^2}\cdot 3\right] \\
&= 3(x+1)\,[\arctan(3x)]^{x+1-1}\,\frac{1}{1+(3x)^2} + [\arctan(3x)]^{x+1}\ln|\arctan(3x)| \\
&= 3(x+1)\,[\arctan(3x)]^{x}\,\frac{1}{1+(3x)^2} + [\arctan(3x)]^{x+1}\ln|\arctan(3x)|.
\end{aligned}$$

Question 2:

To calculate this integral, combine the numerator and denominator terms one at a time:

$$\begin{aligned}
\int_1^\infty \frac{x+\sqrt{x}}{x^{5/2}}dx &= \int_1^\infty \left(x^{-3/2} + x^{-2}\right)dx \\
&= \lim_{T\to\infty}\int_1^T \left(x^{-3/2} + x^{-2}\right)dx \\
&= \lim_{T\to\infty}\left[-2x^{-1/2} - x^{-1}\right]_1^T dx \\
&= \lim_{T\to\infty}\left[-2(T)^{-1/2} - (T)^{-1} + 2 + 1\right] \\
&= 3 - \lim_{T\to\infty}\left[2(T)^{-1/2} + (T)^{-1}\right] \\
&= 3.
\end{aligned}$$

Question 3:

Begin by substituting $x = 0$:

$$\lim_{x\to 0} \frac{\cos 5(0) - \cos 7(0)}{(0)^2} = \frac{1-1}{0} = \frac{0}{0}.$$

Since this is an indeterminate form, we apply L'Hopital's Rule:

$$\lim_{x\to 0} \frac{-5\sin(5x) + 7\sin(7x)}{2x} = \frac{-5(0) + 7(0)}{0} = \frac{0}{0}.$$

Apply L'Hopital again:

$$\lim_{x\to 0} \frac{-25\cos(5x) + 49\cos(7x)}{2} = \frac{-25+49}{2} = 12.$$

Question 4: Write the general form of the trapzoid rule evaluated with $\delta x = 3$ and 7 intervals:

$$\begin{aligned} \mathrm{TRAP}_7(f) &= \frac{3}{2}\left[f(0) + 2f(3) + 2f(6) + 2f(9) + 2f(12) + 2f(15) + 2f(18) + f(21)\right] \\ &= 1.5\left[0 + 2(2.4) + 2(3.4) + 2(4) + 2(3.7) + 2(3.3) + 2(2.3) + 0\right] \\ &= 57.3. \end{aligned}$$

Question 5:

To calculate this integral, we use Partial Fractions:

$$\frac{3x-1}{x^2+x-6} = \frac{3x-1}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{x-2}.$$

Then $A(x-2) + B(x+3) = 3x - 1$ so that $A + B = 3$ and $-2A + 3B = -1$. This has solution $A = 2$ and $B = 1$. Thus:

$$\begin{aligned} \int_0^1 \frac{3x-1}{x^2+x-6} &= \int_0^1 \left(\frac{2}{x+3} + \frac{1}{x-2}\right) dx \\ &= 2\ln|x+3| + \ln|x-2| \Big|_0^1 \\ &= 2\ln|4| + \ln|1| - 2\ln|3| - \ln|2| \\ &\approx -0.1177. \end{aligned}$$

Question 6:

This is a tricky Integration by Parts: choosing dv is the most important step. The best way to arrive at the correct answer is to realize you **must** use the $(x^6+1)^{-5/2}$ piece. To actually be able to integrate this piece, consider a "fake" Substitution Integral using $k = x^6+1$, then $dk = 6x^5$. So whatever we have as integrand dv, we need to include a x^5 piece to cancel the derivative of the (x^6+1) bit. Thus:

$$\begin{array}{ll} u = x^6 & v = -\tfrac{2}{3}(x^6+1)^{-3/2}\tfrac{1}{6} \\ du = 6x^5 & dv = x^5(x^6+1)^{-5/2}dx. \end{array}$$

(Note that v is computed using Substitution, as above, and u is the rest of the original integrand)

Then:

$$\begin{aligned} \int \frac{x^{11}}{(x^6+1)^{5/2}}dx &= -\frac{2}{3}(x^6+1)^{-3/2}\frac{1}{6}x^6 + \int \frac{2}{3}x^5(x^6+1)^{-3/2}dx \\ &= -\frac{2}{3}(x^6+1)^{-3/2}\frac{1}{6}x^6 + \frac{2}{3}(x^6+1)^{-1/2}(-2)\frac{1}{6} \\ &= -\frac{1}{9}x^6(x^6+1)^{-3/2} - \frac{2}{9}(x^6+1)^{-1/2}. \end{aligned}$$

Question 7:

The setup for this problem is key. Draw a triangle representing the cross-section of the pyramid at its exact center, at right angles to two of the sides (so the cross-section which has the largest area while still being parallel to the two perpendicular sides). This triangle has height $146m$ and base $230m$. Using similar triangles, at a height h above the ground,

$$\frac{146}{230/2} = \frac{146-h}{s/2}$$

where s is the side length of the horizontal slice through the pyramid that we are interested in, at height h. This gives us a relationship between s and h: $s = 230 - 230/146h$. Now, write the mass of a slice as:

$$\text{mass} = \left(230 - \frac{230}{146}h\right)^2 \cdot 2360\Delta h.$$

Then the work is the product of the height lifted, the mass, and gravity:

$$\begin{aligned} W &= \int_0^{146} \left(230 - \frac{230}{146}h\right)^2 \cdot 2360 \cdot 9.8 h dh \\ &= 2360 \cdot 9.8 \int_0^{146} \left(230^2 - \frac{2 \cdot 230^2}{146}h + \frac{230^2}{146^2}h^2\right) h dh \\ &= 2360 \cdot 9.8 \left[\frac{230^2}{2}h^2 - \frac{2 \cdot 230^2}{3 \cdot 146}h^3 + \frac{230^2}{4 \cdot 146^2}h^4\right]_{h=0}^{h=146} \\ &= 2.173 \times 10^{12} J. \end{aligned}$$

Question 8:

When drawing this shape, it is a hyperbola with base at $(0, 3)$ and points of intersection (with $x = 4$) at $(4, 5)$ and $(4, 1)$. Rotate this about the line $x = -1$. Using cylindrical shells, we have $r = x + 1$ on $x \in [0, 4]$, and the height of the shell at radius r is $h = \sqrt{x} + 3 - (-\sqrt{x} + 3) = 2\sqrt{x}$. Then:

$$\begin{aligned} V &= \int_0^4 2\pi(x+1)2\sqrt{x} dx \\ &= 4\pi \int_0^4 \left(x^{1.5} + x^{0.5}\right) dx \\ &= 4\pi \left[\frac{2}{5}x^{2.5} + \frac{2}{3}x^{1.5}\right]_{x=0}^{x=4} \\ &= 4\pi \left(\frac{2}{5}(4)^{2.5} + \frac{2}{3}(4)^{1.5}\right) \\ &= \frac{1088}{15}\pi \approx 228. \end{aligned}$$

Question 9: This question is quite hard to parse. Take your time, and make sure you understand the question. Label the diagram with all the distances.

(a) We cannot solve the problem by multiplying the area of the window by the pressure because the pressure is not constant across the entire window. The bottom of the window gets more pressure than the top.

(b) Thin horizontal strips are suitable because pressure is constant in slices across the tank, being dependent only on depth and not any other position.

(c) The shape of the window is $y = -x^2$, where y is the distance below the top of the window. Define our strip to be y_a m below the surface of the water, then each horizontal strip has area $2\sqrt{y}\Delta y$, and the pressure on this strip is $\rho g(y+2) = 1000(9.8)(y+2)$. Then:

$$\begin{aligned} F &= \int_0^1 2\sqrt{y} \cdot 1000 \cdot 9.8 \cdot (y+2) dy \\ &= 9800 \cdot 2 \int_0^1 \left(y^{1.5} + 2\sqrt{y}\right) dy \\ &= 9800 \cdot 2 \left[\frac{2}{5} y^{2.5} + \frac{4}{3} y^{1.5}\right]_{y=0}^{y=1} \\ &= 9800 \cdot 2 \left[\frac{2}{5} + \frac{4}{3}\right] = 3.4 \times 10^4 N. \end{aligned}$$

Question 10:

This is a separable differential equation:

$$\begin{aligned} \int \frac{dS}{1 - 0.2S} &= \int \frac{dT}{20} \\ -5 \ln |1 - 0.2S| &= \frac{1}{20} t + C \\ \ln |1 - 0.2S| &= -\frac{1}{4} t + C \\ 1 - 0.2S(t) &= Ce^{-0.25t} \\ S(t) &= 5 - Ce^{-0.25t} \\ &\quad \text{(apply initial condition)} \\ S(0) &= 5 - Ce^0 \qquad C = -10. \\ S(t) &= 5 + 10e^{-0.25t}. \end{aligned}$$

Now, answer the question: after 3 minutes, there will be

$$S(3) = 5 + 10e^{-0.75} \approx 9.7\text{kg}.$$