APSC 171
2009 December Exam
Solutions

Question 1:

(a) Apply the Quotient Rule:

$$\frac{d}{dx}\frac{\cos(x^2+1)}{\sin^2(x)+1} = \frac{-\sin(x^2+1)(2x)[\sin^2(x)+1] - \cos(x^2+1)[2\sin(x)\cos(x)]}{\left(\sin^2(x)+1\right)^2}.$$

(b) Apply the general power rule, which involves use of the exponential trick:

$$\frac{d}{dx}\left(\arctan(x)\right)^{x^3} = \frac{d}{dx}e^{x^3\ln[\arctan(x)]}.$$

Now, differentiate the exponential function using the exponential and chain rules:

$$\begin{aligned} e^{x^3\ln[\arctan(x)]}\frac{d}{dx}\left(x^3\ln[\arctan(x)]\right) &= e^{x^3\ln[\arctan(x)]}\left[3x^2\ln[\arctan(x)] + x^3\frac{1}{\arctan(x)}\frac{1}{1+x^2}\right] \\ &= 3x^2\left(\arctan(x)\right)^{x^3}\ln[\arctan(x)] + \frac{x^3}{1+x^2}\left(\arctan(x)\right)^{x^3}\left(\arctan(x)\right)^{-1} \\ &= 3x^2\left(\arctan(x)\right)^{x^3}\ln[\arctan(x)] + \frac{x^3}{1+x^2}\left(\arctan(x)\right)^{x^3-1} \end{aligned}$$

Question 2:

To calculate this integral, combine the numerator/denominator pieces to make:

$$\begin{aligned} \int_1^4 \frac{\sqrt{x}+x^2+1}{3x\sqrt{x}}dx &= \int_1^4\left(\frac{1}{3}x^{-1}+\frac{1}{3}x^{1/2}+\frac{1}{3}x^{-1.5}\right)dx \\ &= \left[\frac{1}{3}\ln|3x| + \frac{2}{9}x^{1.5} - \frac{2}{3}x^{-0.5}\right]_{x=1}^{x=4} \\ &= \frac{1}{3}\ln|4| + 1 \end{aligned}$$

where the final step combines all the natural logs in the answer using basic properties.

Question 3:

Use substitution for this integral, letting $u = \cos(x)$ and $du = -\sin(x)dx$ so that $dx = -\dfrac{du}{\sin(x)}$. Then:

$$\begin{aligned}
\int_0^{\pi} \frac{\sin(x)}{1+\cos^2(x)}dx &= -\int_{x=0}^{x=\pi} \frac{\sin(x)}{1+u^2}\frac{du}{\sin(x)} \\
&= -\int_{x=0}^{x=\pi} \frac{1}{1+u^2}du \\
&= -\left[\arctan(u)\right]_{x=0}^{x=\pi} \\
&= -\left[\arctan(\cos(x))\right]_{x=0}^{x=\pi} \\
&= -\arctan(-1) + \arctan(1) = \pi/4 + \pi/4 \\
&= \frac{\pi}{2}.
\end{aligned}$$

Question 4:

(a) The increase in **mass** of salt will be the scaled version of the concentration (into mass per kilogram instead of mass per million kilograms) multiplied by the number of kilograms of water. Thus:

$$\text{mass of salt } = \int_0^{12} \frac{s(t)}{1000000} f(t)dt.$$

(b) The integral of $f(t)$ is the total amount of water flowing into the holding tank from time $t = 1$ to time $t = x$. The derivative of this becomes the rate of water flow at time $t = x$, $f(x)$.

(c) Simpson's Rule applied on the "various times" available to us, using the integral presented in part (a), with the 1/1000000 factored out, is:

$$\begin{aligned}
\text{SIMP}_6 = \frac{2}{3}\frac{1}{1000000}&\left[s(0)f(0) + 4s(2)f(2) + 2s(4)f(4) + 4s(6)f(6) + 2s(8)f(8)\right. \\
&\left. + 4s(10)f(10) + s(12)f(12)\right] \\
= \frac{2}{3000000}&\left[70.5 + 4(83.3) + 2(80) + 4(67.2) + 2(60) + 4(45.9) + 58.8\right] \times 10^2 \\
= 0.07966 &\approx 0.08\text{kg}.
\end{aligned}$$

Question 5:

(a) If we were not computing kinetic energy, but instead gravitational potential energy, the two methods would be equivalent. In the case of kinetic, we have to have a uniform distance from the axis of rotation for each infintesimal piece – this implies a circle centered on the axis of rotation. But this has to hold for the entire axis, which implies a cylinder.

(b) Find the shell volume first:

$$2\pi x \ln(x)\Delta x$$

and then convert to mass by multiplying by 0.8 kg/m^3. Then, substitute into the expression provided:

$$E = \frac{1}{2}(0.8)2\pi x \ln(x)\Delta x \omega^2 x^2.$$

(c) Now, integrate as x runs from $x = 1$ to $x = e$:

$$\int_{x=1}^{x=e} 0.4 \cdot 2\pi x \ln(x)\omega^2 x^2 dx = 0.8\pi\omega^2 \int_1^e x^3 \ln x dx.$$

Now, use integration by parts, setting $u = \ln(x)$ and $dv = x^3 dx$ which gives $du = 1/x dx$ and $v = \frac{1}{4}x^4$. Then:

$$\begin{aligned} 0.8\pi\omega^2 \int_1^e x^3 \ln x dx &= 0.8\pi\omega^2 \left[\frac{1}{4}x^4 \ln(x)\Big|_1^e - \int_1^e \frac{1}{4}x^3 dx\right] \\ &= 0.8\pi\omega^2 \left[\frac{1}{4}e^4 - \left(\frac{1}{16}x^4\Big|_1^e\right)\right] \\ &= 0.8\pi\omega^2 \left[\frac{3}{16}e^4 + \frac{1}{16}\right] \\ &\approx 25.9\omega^2. \end{aligned}$$

Question 6:

Begin by sketching a graph (shown in Figure).

Now, the easiest way to solve this volume integral is to use cylindrical shells at radius $x + 3$ for $x \in [0, 1]$. The cylinders have individual volume

$$V_i = 2\pi(x+3)\frac{1}{x^2 + 6x + 5}\Delta x.$$

To find the total volume, integrate this from $x = 0$ to $x = 1$:

$$V = \int_0^1 2\pi(x+3)\frac{1}{x^2 + 6x + 5}dx.$$

To solve this, use Partial Fractions to expand the denominator, which factors into $(x+5)(x+1)$:

$$\frac{x+3}{(x+5)(x+1)} = \frac{A}{x+5} + \frac{B}{x+1}$$

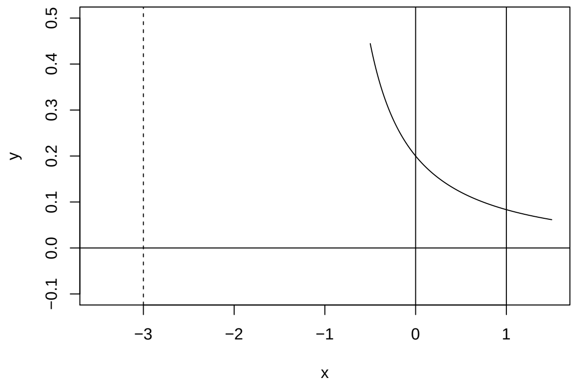

Figure 1: Figure for Q6.

so that $A(x+1)+B(x+5)=x+3$ implying $A=B=1/2$. Then:

$$\begin{aligned} V &= \int_0^1 2\pi(x+3)\frac{1}{x^2+6x+5}dx = 2\pi\int_0^1 \left(\frac{1}{2(x+5)}+\frac{1}{2(x+1)}\right)dx \\ &= \pi\left[\ln|x+5|+\ln|x+1|\right]_{x=0}^{x=1} \\ &= \pi\left[\ln|6|+\ln|2|-\ln|5|-\ln|1|\right] \\ &= \pi\ln\left|\frac{12}{5}\right| \approx 2.75. \end{aligned}$$

Question 7:

This is a bit of a tricky problem. The key is to recognize that the work is computed for the **original** position of each slab of fuel, not the bottom of the tank where the fuel actually exists. You can think of this as the fuel losing potential energy as the fuel level in the tank falls (negative work), and then gaining energy again as it rises up the hose (positive work). In the end, the work done is only the difference in the original height and the tanker truck height.

Begin by considering the cross-sectional area, which will be a circular slab of height Δh and area πr^2. Now, for a given height of fuel in the tank $h \in [0,2]$, the radius of the corresponding slice is $r=\sqrt{4-(2-h)^2}$. Thus, write the work formula:

$$\begin{aligned}
W &= \int_{h=0}^{h=2} Fhdh = \int_{h=0}^{h=2} mghdh = \int_{h=0}^{h=5} m(9.8)(5-h)dh \\
&= \int_{h=0}^{h=2} 900 \cdot \pi r^2 (9.8)(5-h)dh \\
&= 900\pi(9.8) \int_{h=0}^{h=2} \left(4-(2-h)^2\right)(5-h)dh \\
&= 9.8(900)\pi \int_{h=0}^{h=2} \left(4-4+4h-h^2\right)(5-h)dh \\
&= 9.8(900)\pi \int_{h=0}^{h=2} \left(20h-9h^2+h^3\right)dh \\
&= 9.8(900)\pi \left[10h^2-3h^3+\frac{1}{4}h^4\right]_{h=0}^{h=2} \\
&= 5.54\times 10^5 J.
\end{aligned}$$

Question 8:

(a) We have that $T_a = 180°$, so write

$$\frac{dT}{dt} = -k(T(t)-180).$$

(b) Now, $T(0) = 20°$ so (you should have this solution memorized!)

$$T(t) = T_a + (T_0 - T_a)e^{-kt} = 180 + (20-180)e^{-kt}.$$

Use the information provided ($T(1) = 30$):

$$T(1) = 30 = 180 - 160e^{-k}$$

which gives that $k = -\ln|150/160| = 0.065$. Thus:

$$T(2) = 180 - 160e^{-0.065(2)} \approx 39.5°C.$$

Question 9:

(a) Recall that speed is the magnitude of velocity, so

$$\vec{v}(t) = \langle 3t^2, 2t, 1\rangle \qquad t\in\mathbb{R}.$$

So compute the magnitude:

$$|\vec{v}(t)| = \sqrt{(3t^2)^2+(2t)^2+1}\Big|_{t=-1} = \sqrt{9+4+1} = \sqrt{14}.$$

(b) Set the two trajectories equal to each other and solve, keeping the variables different (since it's just the trajectories, not the actual spacecraft colliding):

$$\begin{aligned} t^3 &= u^2 - 2u \\ t^2 &= u \\ t &= u - 2. \end{aligned}$$

Using the last relationship, substitute into the second relationship:

$$(u-2)^2 = u \qquad \Rightarrow \qquad u^2 - 4u + 4 - u = 0$$

which has solutions $u = 1, 4$, which have corresponding $t = -1, 2$. Check the first relationship: $u = 1 \Rightarrow t = -1$ (matches) and $u = 4 \Rightarrow t = 2$ (matches). Thus we have two intersections of the two trajectories, one at the point $\langle -1, 1, -1 \rangle$ and the second at the point $\langle 8, 4, 2 \rangle$.

(c) To calculate this integral, set up the limit:

$$\begin{aligned} \int_{-\infty}^{0} \frac{1}{(8-x)^{7/3}} dx &= \lim_{T \to -\infty} \int_{T}^{0} (8-x)^{-7/3} dx \\ &= \lim_{T \to -\infty} \left[\frac{3}{4} (8-x)^{-4/3} \right]_T^0 \\ &= \lim_{T \to -\infty} \left[\frac{3}{4} (8)^{-4/3} - \frac{3}{4} (8-T)^{-4/3} \right] \\ &= \frac{3}{64} - \frac{3}{4} \left(\lim_{T \to -\infty} \left[\frac{1}{8-T} \right]^{4/3} \right) \\ &= \frac{3}{64}. \end{aligned}$$