QUEEN'S UNIVERSITY
APSC 171J – Assignment 6
Wesley Burr
Due: Not for Grade (do before Final!)

INSTRUCTIONS

- This assignment includes three problems to help you prepare for the final exam.
- As announced in class, there will be a differential equation problem on the final. Since we did not have time to cover the material **and** include it on assignment 5, these questions are intended to fill in that gap and give you a chance to practice.
- This file is the posted solution.

FOR INSTRUCTOR'S USE ONLY		
Question	Mark Available	Received
1	4	
2	4	
3	4	
TOTAL	12	

1. [4 marks] Suppose that a cup of coffee is left on a table. A thirsty mathematician finds this cup of (cold) coffee and measures its temperature to be $32°$ C. The room the cup was in is kept at a constant $20°$ C. Considering the coffee too cold to drink, the mathematician walks away. Three hours later, he realizes the cup is still sitting on the table, having dropped in temperature to $27°$ C. If coffee comes out of the coffee maker at $60°$ C, how many hours ago was the cup left on the table?

We begin this problem by writing the differential equation that corresponds to Newton's Law of Cooling:

$$\frac{dT}{dt} = k\left(T(t) - T_a\right)$$

where we have chosen to use the $k < 0$ variation of the formulation. Now, solving this differential equation:

$$\begin{aligned}
\frac{dT}{T - T_a} &= kdt \\
\int \frac{dT}{T - T_a} &= \int kdt \\
\ln|T - T_a| &= kt + C \\
e^{\ln|T - T_a|} &= e^{kt+C} \\
|T - T_a| &= e^C e^{kt} \\
T - T_a &= \pm C_2 e^{kt} \\
T(t) &= T_a \pm C_2 e^{kt} \\
&= T_a + C_3 e^{kt}.
\end{aligned}$$

If you happen to remember the form of Newton's Law of Cooling, you may remember that it involves a $T_0 = T(0)$: the temperature of the object at time 0. Substituting $t = 0$ into the derived equation:

$$T(0) = T_0 = T_a + C_3 e^{k(0)}$$

so $C_3 = T_0 - T_a$. Thus:

$$T(t) = T_a + (T_0 - T_a)e^{kt}.$$

Now that we have derived the equation, we use the available pieces of information to solve for the constants. We know $T_a = 20°$ C (given), and we know $T_0 = 60°$ C (since coffee apparently comes out of the machine at $60°$ C). We know that $T(t_i) = 32°$ C, where t_i is the time (in hours) from when the coffee was poured to the time the mathematician finds it on the table. And, finally, we know that $T(t_i + 3) = 27°$ C.

There are several approaches that we can use now to find k. We will show two of them. The first is a frame of reference change, where we let $t_i = 0$. This method then changes T_0 from $T_0 = 60$ to $T_0 = 32$ (temporarily), so

$$T(t_i) = 32 = T(0) = 20 + (32 - 20)e^{k(0)}$$

(rather obviously), **and**

$$T(t_i + 3) = 27 = 20 + (32 - 20)e^{k(3)}$$

so that

$$7 = 12e^{3k}$$

or

$$\ln\left|\frac{7}{12}\right| = 3k$$

so that $k = -0.1797$. The second method is similar, but does not use the reference frame change, instead taking the difference between the two known points. Write

$$T(t_i) = 32 = 20 + (60 - 20)e^{k(t_i)} \qquad \Rightarrow \qquad \frac{12}{40} = e^{k(t_i)}$$

and

$$T(t_i + 3) = 27 = 20 + (60 - 20)e^{k(t_i+3)} \qquad \Rightarrow \qquad \frac{7}{40} = e^{k(t_i+3)}.$$

Then, simplifying both left hand sides to 1, write

$$1 = \frac{40}{12}e^{kt_i} = \frac{40}{7}e^{k(t_i+3)}$$

so that

$$\frac{40}{12}\frac{7}{40} = e^{k(t_i+3)}e^{-kt_i} = e^{kt_i+3k-kt_i} = e^{3k}$$

and thus

$$\ln\left|\frac{7}{12}\right| = 3k$$

which gives exactly the same $k = -0.1797$.

Now that we have k, T_0 and T_a, return to the question given. Asking how long ago the cup of coffee was left on the table is equivalent to asking what t_i is. Thus:

$$T(t_i) = 32 = 20 + (60 - 20)e^{-0.1797t_i}$$

so that

$$\frac{12}{40} = e^{-0.1797t_i}$$

or

$$\ln\left|\frac{12}{40}\right| = -0.1797t_i \qquad \Rightarrow \qquad 6.6999 = 6.7 \text{ hours.}$$

Thus, the cup of coffee was left on the table 6.7 hours before the mathematician first found it, or 9.7 hours "ago" (if time is being measured from when the mathematician measures the temperature of the cup the second time).

Final Answer:

The cup of coffee was left on the table 6.7 hours before the mathematician first found it.

2. [4 marks] Suppose a corpse is discovered in a motel room at midnight, and its temperature is $80°$ F. The room is kept at a constant $60°$ F. Two hours later the temperature of the corpse (which has not been moved) has dropped to $67°$ F. Find the time of death. Note that the human body usually sits at $98.6°$ F when in good health.

We will not show the derivation of the $T(t)$ equation again, as it is the same for all of these questions. Instead, assume we write down the two equations:

$$\frac{dT}{dt} = -k(T(t) - T_a) \qquad \text{and } T(t) = T_a + (T_0 - T_a)e^{-kt}$$

where we use the alternative form e^{-kt} for this example. Now, we are given $T(t_i) = 80°$ F, $T_a = 60°$ F, and $T(t_i + 2) = 67°$ F. We are also provided with the piece of information that $T_0 = T(0) = 98.6°$ F.

Begin by finding k using the two equations:

$$T(t_i) = 80 = 60 + (98.6 - 60)e^{-kt_i} \qquad \text{and} \qquad T(t_i + 2) = 67 = 60 + (98.6 - 60)e^{-k(t_i+2)}$$

which simplifies to

$$\frac{20}{38.6} = e^{-kt_i} \qquad \frac{7}{38.6} = e^{-kt_i - 2k}$$

Rearranging and setting these two equal to each other, we obtain

$$\begin{aligned} 1 &= \frac{38.6}{20}e^{-kt_i} = \frac{38.6}{7}e^{-kt_i-2k} \\ \frac{7}{20} &= e^{-kt_i-2k}e^{kt_i} \\ \ln\left|\frac{7}{20}\right| &= \ln\left|e^{-2k}\right| \\ -1.04982 &= -2k \\ k &= 0.5249. \end{aligned}$$

Now, we want to find t_i:

$$\begin{aligned} T(t_i) &= 80 = 60 + (98.6 - 60)e^{-0.5249(t_i)} \\ \ln\left|\frac{20}{38.6}\right| &= -0.5249t_i \\ t_i &= 1.253. \end{aligned}$$

Thus the person died 1.253 hours before they were found.

Final Answer:

Our analysis shows that the person died 1.253 hours before they were found.

3. [4 marks] Newton's Law of Cooling says that the rate at which an object heats or cools is proportional to the difference between the temperature of the object and its surroundings. Translate this into a mathematical equation. If a cup of coffee at $100°$ C is placed on a table in a room whose temperature is $20°$ C, and it is observed at a later point that the coffee is cooling at a rate exactly half that of its initial cooling rate (i.e., $0.5 \times dT(0)$), calculate the temperature of the coffee at that point.

We begin with the mathematical equation: the rate of change of temperature of an object dT/dt is proportional to the difference between $T(t)$, the temperature of the object at time t, and the ambient temperature, T_a. Thus:

$$\frac{dT}{dt} \propto T(t) - T_a$$

which is equivalent to

$$\frac{dT}{dt} = k\left(T(t) - T_a\right).$$

We are given that $T(0) = 100°$ C, $T_a = 20°$ C, and that $\frac{dT}{dt}(t_i) = 0.5 \cdot \frac{dT}{dt}(0)$. Thus:

$$\frac{dT}{dt}(0) = k\left(T(0) - T_a\right) = k\left(100 - 20\right) = 80k.$$

Now,

$$\frac{dT}{dt}(t_i) = 0.5\frac{dT}{dt}(0) = 0.5(80k) = 40k.$$

But:

$$\frac{dT}{dt}(t_i) = k\left(T(t_i) - T_a\right) = k\left(T(t_i) - 20\right)$$

so

$$0.5(80k) = k\left(T(t_i) - 20\right)$$

or

$$40 = T(t_i) - 20 \qquad \Rightarrow \qquad T(t_i) = 60°\text{C}.$$

Thus, when the coffee is $60°$C it will be cooling at exactly half the rate that it cooled at when it was $100°$C.

Final Answer:

The coffee will be $60°$C when it is cooling at exactly half the rate it cooled at when originally poured.