QUEEN'S UNIVERSITY
APSC 171J – Assignment 5: Solutions
Wesley Burr
Due: February 14, 2013

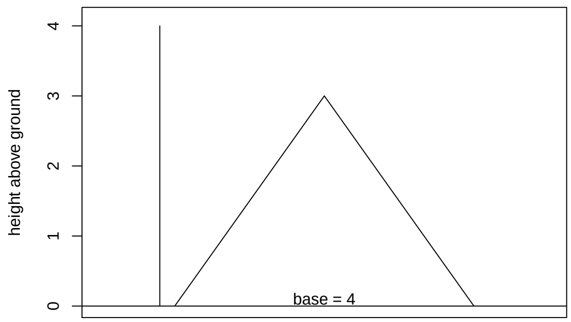
INSTRUCTIONS

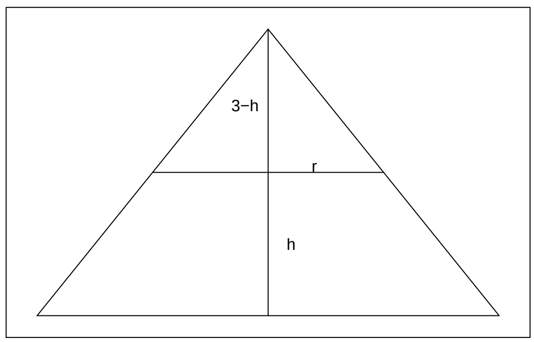
- This assignment is due in-class (4:30-5:20pm) Thursday, February 14, 2013.
- Answer all questions, writing clearly on the sheets provided. **You must print this file and hand in a carefully stapled copy!** Unstapled assignments will not be accepted.
- One mark in each question is for **complete** (and mostly correct) work shown
- The second mark is for a **fully** correct solution, which **must** be placed in the box provided
- If more than two marks are provided for a question, expect the question to require more work and logically divide into sections, each of which will be worth a mark.
- Whenever possible, simplify your solution.
- There are no part marks: you will receive integer marks only for each question.
- **This assignment is out of 26, but has 27 marks. The last mark is bonus.**

FOR INSTRUCTOR'S USE ONLY		
Question	Mark Available	Received
1	4	
2	3	
3	4	
4	4	
5	4	
6	4	
7	4	
TOTAL	26	

1. [4 marks] A pile of gravel in the shape of a cone (with height 3m and base of radius 2m) needs to be lifted over a 4m fence. If the density of gravel is $\rho = 1600$ kg/m^3, how much work is required to lift this gravel up?

Start by drawing a diagram of the pile of gravel and the fence: and then start expanding





our sequence of equations, beginning with the cross-sectional volume of a slice. Before we do this, write the similar triangle relationship:

$$\frac{3}{2} = \frac{3-h}{r} \qquad \Rightarrow \qquad r = \frac{1}{3}(6-2h).$$

So:

$$V_{slice} = \pi r^2 \Delta h = \pi\left(\frac{1}{3}(6-2h)\right)^2 \Delta h$$
$$m_{slice} = V \cdot \rho = \pi\left(\frac{1}{3}(6-2h)\right)^2 \Delta h\,(1600)$$
$$F_{slice} = mg = \pi\left(\frac{1}{3}(6-2h)\right)^2 \Delta h\,(1600)\,(9.81)$$
$$W_{slice} = Fd = \pi\left(\frac{1}{3}(6-2h)\right)^2 \Delta h\,(1600)\,(9.81)\,(4-h)$$

where the final inclusion of $d = 4-h$ comes from our definition of h as coming from the ground up. If we consider $h = 0$, we lift this slice 4m; if we consider $h = 3$m, then we lift that slice 1m. The linear function which represents these two points is $d = 4-h$. Finally, we integrate the work for a slice over many slices, $h = 0$ through $h = 3$ to obtain:

$$\begin{aligned}
W &= \int_0^3 \pi \left(\frac{1}{3}(6-2h)\right)^2 (1600)\,(9.81)\,(4-h)\,dh \\
&= \frac{1600(9.81)\pi}{9} \int_0^3 (6-2h)^2(4-h)dh \\
&= 5478.933 \int_0^3 \left(36 - 24h + 4h^2\right)(4-h)\,dh \\
&= 5478.933 \int_0^3 \left(144 - 132h + 40h^2 - 4h^3\right) dh \\
&= 5478.933 \left[144h - 66h^2 + \frac{40}{3}h^3 - \frac{4}{4}h^4\right]_{h=0}^{h=3} \\
&= 5478.933 \left[144(3) - 66(9) + \frac{40}{3}(27) - (81)\right] \\
&= 5478.933\,[117] \\
&\approx 6.41 \times 10^5 J.
\end{aligned}$$

Final Answer:

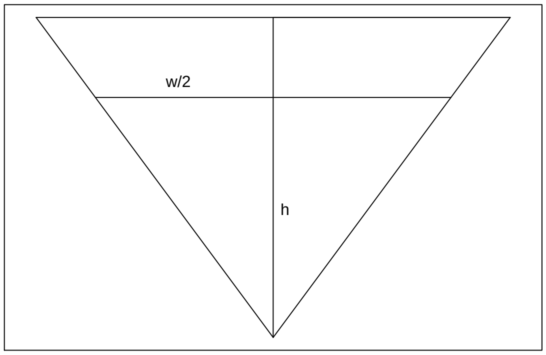
The work required to lift the pile of gravel over the fence is 6.41×10^5 J.

2. [3 marks] Water is running into a trough whose length is 2m and whose cross-section is an upside-down right-angled isosceles triangle with base 40 cm. If the water enters the trough at a rate of 0.2m^3 per minute, and if the water level is 10 cm from the bottom, how fast is the water level rising? *Note: this is not a work problem!*

Begin by writing the volume of the water in the trough:

$$V = \frac{1}{2} b \cdot h \cdot l$$

where we use the area of a triangle $1/2bh$ and the length. Then, draw a cross-sectional area: and from this obtain the relationship



$$\frac{b}{h} = \frac{0.4}{0.2} \qquad \Rightarrow \qquad b = 2h.$$

Then:

$$\begin{aligned} V &= \frac{1}{2} b \cdot h \cdot l = \frac{1}{2}(2h)\, h \cdot 2 \\ &= 2h^2 \\ &\text{and so} \\ \frac{dV}{dt} &= 4h \cdot \frac{dh}{dt} \\ \frac{dV}{dt}\frac{1}{4h} &= h' = \left(0.2\text{m}^3/\text{min}\right)\frac{1}{4\,(0.1\text{m})} \\ 0.5\text{m}/\text{min} &= h' = \frac{dh}{dt}. \end{aligned}$$

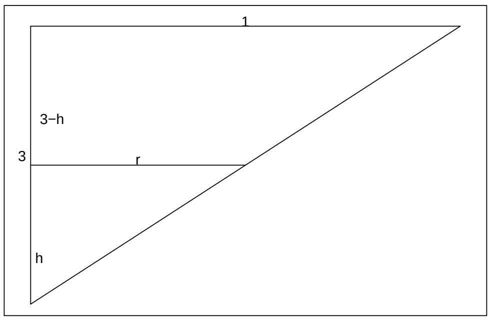
Thus the water level is rising at 0.5 m/min at the moment described.

Final Answer:

The water level is rising at 0.5 m/min.

3. [4 marks] A conical tank has height 3m and a radius of 1m at the top. If the bottom of the tank is 2m above the ground, find the work done in filling the tank with water that originates at ground level.

Begin by examining the similar triangles for the inverted cone shape:



$$\frac{1}{3} = \frac{r}{h} \qquad \Rightarrow \qquad r = \frac{1}{3}h.$$

Then, find the cross-sectional volume and continue the process:

$$\begin{aligned}
V &= \pi r^2 \Delta h = \pi \left(\frac{1}{3}h\right)^2 \Delta h = \frac{\pi}{9}h^2 \Delta h \\
m &= V \cdot \rho = \frac{1000\pi}{9}h^2 \Delta h \\
F &= m \cdot g = \frac{9810\pi}{9}h^2 \Delta h \\
W_{\text{slice}} &= F \cdot d = \frac{9810\pi}{9}h^2 \Delta h \,(h+2)\,.
\end{aligned}$$

Then, integrate over the water in the tank, from $h = 0$ through to $h = 3$:

$$\begin{aligned}
W &= \int_0^3 W_{\text{slice}} = \int_0^3 \frac{9810\pi}{9}h^2(h+2)dh \\
&= \frac{9810\pi}{9}\int_0^3 \left(h^3 + 2h^2\right) dh \\
&= \frac{9810\pi}{9}\left[\frac{1}{4}h^4 + \frac{2}{3}h^3\right]_{h=0}^{h=3} \\
&= \frac{9810\pi}{9}\left[\frac{1}{4}(3)^4 + \frac{2}{3}(3)^3 - 0 - 0\right] \\
&= \frac{9810\pi}{9}[38.25] \\
&= 130,980.74 \approx 1.31 \times 10^5 J.
\end{aligned}$$

Final Answer:

The work required to fill the tank is $1.31 \times 10^5 J$.

4. [4 marks] Water is to be pumped out of a lake into a triangular prism-shaped trough, whose width at the top is 2m, whose depth at the center is 1m, and whose length is 4m. Suppose the bottom of the trough is 2m above the lake. How much work is required to fill to the trough full of lake water?

Since the cross-section of this trough is a triangle, our "similar triangles" will be the entire cross section (base 2, height 1) and the cross-section of water partially filling it (base b, height h). Thus:

$$\frac{2}{1} = \frac{b}{h} \qquad \Rightarrow \qquad b = 2h.$$

Write the usual sequence of equations:

$$\begin{aligned} V_{\text{slice}} &= b \cdot l \cdot \Delta h = (2h)(4)\Delta h \\ m &= V \cdot \rho = 1000 \cdot 8h\Delta h \\ F &= mg = 9810 \cdot 8h\Delta h \\ W_{\text{slice}} &= 9810 \cdot 8h\Delta h \cdot (2+h) \end{aligned}$$

and then we integrate the work for a single slice over all slices, from $h = 0$ to $h = 1$:

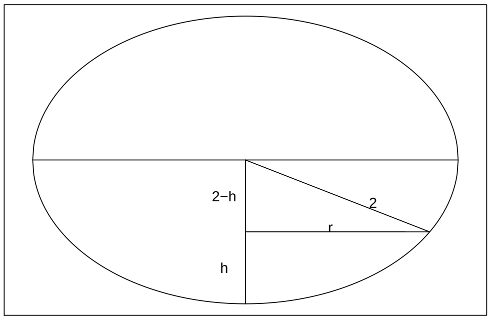
$$\begin{aligned} W &= \int_0^1 W_{\text{slice}} \\ &= \int_0^1 9810 \cdot 8h \cdot (2+h) dh \\ &= 8(9810) \int_0^1 \left(2h + h^2\right) dh \\ &= 8(9810) \left[h^2 + \frac{1}{3}h^3\right]_{h=0}^{h=1} \\ &= 78,480 \left[1 + \frac{1}{3} - 0 - 0\right] \\ &= 104,640J \approx 1.05 \times 10^5 J. \end{aligned}$$

Final Answer:

The work required to fill the trough is 1.05×10^5 J.

5. [4 marks] An empty spherical tank of radius 2m is constructed by the side of a lake in such a way that the vertical distance between the bottom of the water tank and the surface of the lake is 5m. Find the work required to half-fill the tank with lake water.

Begin by drawing a very careful diagram of this spherical tank:



$$(2-h)^2 + r^2 = 2^2 \qquad \Rightarrow r^2 = 4 - (2-h)^2.$$

and then find the cross-sectional volume (which is a disc; circle + thickness):

$$\begin{aligned} V_{\text{slice}} &= \pi r^2 = \pi\left(4 - (2-h)^2\right)\Delta h \\ m &= V\rho = 1000\pi\left(4 - (2-h)^2\right)\Delta h \\ F &= mg = 9810\pi\left(4 - (2-h)^2\right)\Delta h \\ W_{\text{slice}} &= Fd = 9810\pi\left(4 - (2-h)^2\right)\Delta h\,(5+h) \end{aligned}$$

and then integrate the work from $h = 0$ to $h = 2$:

$$\begin{aligned} W &= \int_0^2 W_{\text{slice}} = \int_0^2 9810\pi\left(4-(2-h)^2\right)(5+h)\,dh \\ &= 9810\pi \int_0^2 \left(4 - (4 - 4h + h^2)\right)(5+h)dh \\ &= 9810\pi \int_0^2 \left(20h - h^2 - h^3\right)dh \\ &= 9810\pi\left[10h^2 - \frac{1}{3}h^3 - \frac{1}{4}h^4\right]_{h=0}^{h=2} \\ &= 9810\pi\left[40 - \frac{8}{3} - \frac{16}{4}\right] \\ &= 9810\pi\frac{100}{3} = 1,027,299.93 \\ &\approx 1.03 \times 10^6 J. \end{aligned}$$

Final Answer:

The work required to half-fill the sphere with water is 1.03×10^6 J.

6. [4 marks] Find the points on the ellipse $2x^2 + y^2 = 1$ with the property that the tangent lines at these points go through the point $(2, 0)$.

Begin by parametrizing the ellipse:

$$a^2x^2 + b^2y^2 = r^2 \qquad \Rightarrow \begin{cases} x = \frac{r}{a}\cos(t) = \frac{1}{\sqrt{2}}\cos(t) \\ y = \frac{r}{b}\sin(t) = \sin(t) \end{cases}$$

Then we have our position and tangent vectors as

$$\vec{p} = \langle \frac{1}{\sqrt{2}}\cos(t), \sin(t)\rangle \qquad \vec{p}' = \langle -\frac{1}{\sqrt{2}}\sin(t), \cos(t)\rangle.$$

The vector $\vec{p}'$ has the slope of the tangent line to the ellipse at the point corresponding to time t. We draw a line from $(2, 0)$ to the ellipse, and then look for times t where the two vectors point in the same direction. The first vector is

$$\vec{q} = \langle 2, 0\rangle - \langle \frac{1}{\sqrt{2}}\cos(t), \sin(t)\rangle$$

and the second

$$\vec{p}' = \langle -\frac{1}{\sqrt{2}}\sin(t), \cos(t)\rangle$$

so set these two equal with a proportionality constant:

$$\langle 2 - \frac{1}{\sqrt{2}}\cos(t), -\sin(t)\rangle = \alpha\langle -\frac{1}{\sqrt{2}}\sin(t), \cos(t)\rangle$$

and then recognize that the proportionality constant gives us a way to solve for t:

$$\alpha = \frac{2 - \frac{1}{\sqrt{2}}\cos(t)}{-\frac{1}{\sqrt{2}}\sin(t)} = \frac{-\sin(t)}{\cos(t)}.$$

Cross-multiplying we have

$$2\cos(t) - \frac{1}{\sqrt{2}}\cos^2(t) = \frac{1}{\sqrt{2}}\sin^2(t) \qquad \Rightarrow 2\cos(t) = \frac{1}{\sqrt{2}}(1)$$

Thus we have $\cos(t) = -\frac{1}{2\sqrt{2}}$, which will have solutions in the second and third quadrants. Solving via $\arccos(t)$ on our calculators, we obtain $t = 1.209429$, which has corresponding solution in the third quadrant of $2\pi - 1.209429 = 5.073756$. Plugging these two times into our position vector we obtain

$$\vec{p}(1.209429) = \langle 0.25, 0.9354\rangle$$
$$\vec{p}(5.073756) = \langle 0.25, -0.9354\rangle.$$

Final Answer:

The two points on the ellipse that have tangent lines which go through $\langle 2, 0\rangle$ are $\langle 0.25, 0.9354\rangle$ and $\langle 0.25, -0.9354\rangle$.

7. [4 marks] There are four points P on the ellipse $x^2 + 2y^2 = 8$ such that the normal line at P passes through the point $(0, 1)$. Find the four points. Note that the *normal line* is the line that is perpendicular to the tangent line to a curve at a given point.

This problem is very similar to the previous, except that we are interested in the normal vector instead of the tangent vector. This adds only one step to the process. Begin by parametrizing the ellipse:

$$a^2x^2 + b^2y^2 = r^2 \qquad \Rightarrow \begin{cases} x = \frac{r}{a}\cos(t) = \sqrt{8}\cos(t) \\ y = \frac{r}{b}\sin(t) = \frac{\sqrt{8}}{\sqrt{2}}\sin(t) \end{cases}$$

Then we have our position and tangent vectors as

$$\vec{p} = \langle \sqrt{8}\cos(t), 2\sin(t)\rangle \qquad \vec{p}' = \langle -\sqrt{8}\sin(t), 2\cos(t)\rangle.$$

The normal vector is perpendicular to the tangent vector, so take the negative reciprocal vector as $\vec{r}$:

$$\vec{r}\cdot\vec{p}' = 0 \qquad \Rightarrow \qquad \vec{r} = \langle 2\cos(t), \sqrt{8}\sin(t)\rangle.$$

Now, we use the same setup as before: draw a line from the point of interest to our ellipse.

$$\vec{q} = \langle 0, 1\rangle - \langle \sqrt{8}\cos(t), 2\sin(t)\rangle$$

and the second is $\vec{r}$. Set these two equal with a proportionality constant:

$$\langle 0 - \sqrt{8}\cos(t), 1 - 2\sin(t)\rangle = \alpha\langle 2\cos(t), \sqrt{8}\sin(t)\rangle$$

and then recognize that the proportionality constant gives us a way to solve for t:

$$\alpha = \frac{-\sqrt{8}\cos(t)}{2\cos(t)} = \frac{1 - 2\sin(t)}{\sqrt{8}\sin(t)}.$$

Cross-multiplying we have

$$-8\cos(t)\sin(t) = 2\cos(t) - 4\sin(t)\cos(t) \qquad \Rightarrow \qquad 4\sin(t)\cos(t) = -2\cos(t).$$

The first solution to this equation is $\cos(t) = 0$, and the second is $\sin(t) = -0.5$. The solutions to the first equation are $\pi/2$ and $3\pi/2$, and the second, as $\sin$ is negative, are in quadrants three and four. Thus, $\arcsin(-0.5) = -0.5235988 = 5.759587$, and the second solution is at $\pi + 0.5235988 = 3.665191$. Then we have our four solutions, so plug them into our position vector:

$$\vec{p}(1.570796) = \langle 0.0, 2.0\rangle \qquad \vec{p}(3.665191) = \langle -\sqrt{6}, -1.0\rangle$$
$$\vec{p}(4.712389) = \langle +\sqrt{6}, -1.0\rangle \qquad \vec{p}(5.759587) = \langle 0.0, -2.0\rangle.$$

Final Answer:

The four solutions are positions $\langle 0, 2\rangle$, $\langle 0, -2\rangle$, $\langle -\sqrt{6}, -1\rangle$ and $\langle \sqrt{6}, -1\rangle$.