QUEEN'S UNIVERSITY
APSC 171J – Assignment 3: Solutions
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Due: February 4, 2013

INSTRUCTIONS

- This assignment is due in-class (3:30-4:20pm) Monday, February 4, 2013.
- Answer all questions, writing clearly on the sheets provided. **You must print this file and hand in a carefully stapled copy!** Unstapled assignments will not be accepted.
- One mark in each question is for **complete** (and mostly correct) work shown
- The second mark is for a **fully** correct solution, which **must** be placed in the box provided
- Whenever possible, simplify your solution.
- There are no part marks: you will receive 0, 1 or 2 on each question.

FOR INSTRUCTOR'S USE ONLY		
Question	Mark Available	Received
1	2	
2	2	
3	3	
4	2	
5	2	
6	2	
7	2	
8	2	
9	2	
10	2	
TOTAL	21	

1. [2 marks] Calculate the limit

$$\lim_{x \to -1} \frac{x^2+2x+1}{x^2-x-2}.$$

Begin by substituting the limit:

$$\lim_{x \to -1} \frac{x^2+2x+1}{x^2-x-2} = \frac{(-1)^2+2(-1)+1}{(-1)^2-(-1)-2} = \frac{0}{0}.$$

This is indeterminate, so we can use L'Hopital's Rule:

$$\lim_{x \to -1} \frac{x^2+2x+1}{x^2-x-2} = \lim_{x \to -1} \frac{2x+2}{2x-1} = \frac{2(-1)+2}{2(-1)-1} = \frac{0}{-3} = 0.$$

As this second evaluation is no longer indeterminate, we are done, and the limit is 0.

Final Answer:

$$\lim_{x \to -1} \frac{x^2+2x+1}{x^2-x-2} = 0$$

2. [2 marks] Calculate the limit

$$\lim_{x\to 0} \frac{1}{x} - \frac{1}{e^x - 1}.$$

Begin by evaluating the limits:

$$\lim_{x\to 0} \frac{1}{x} - \frac{1}{e^x - 1} = \frac{1}{0} - \frac{1}{0}.$$

Note that you **cannot** combine $1/0$ and $1/0$ to get $0/0$! This form is **not** indeterminate! However, if we go back to the beginning, we can write

$$\lim_{x\to 0} \frac{1}{x} - \frac{1}{e^x - 1} = \lim_{x\to 0} \frac{1(e^x - 1) - 1(x)}{x\,(e^x - 1)}$$

and if we then take the limit of this, we get

$$\lim_{x\to 0} \frac{1(e^x - 1) - 1(x)}{x\,(e^x - 1)} = \frac{0}{0}$$

which **is** indeterminate. Apply L'Hopital's Rule:

$$\lim_{x\to 0} \frac{1(e^x - 1) - 1(x)}{x\,(e^x - 1)} = \lim_{x\to 0} \frac{e^x - 1}{e^x - 1 + x\,(e^x)} = \frac{0}{0}$$

and as this result is again indeterminate, apply L'Hopital's Rule again:

$$\lim_{x\to 0} \frac{e^x - 1}{e^x - 1 + x\,(e^x)} = \lim_{x\to 0} \frac{e^x}{e^x + x\,(e^x) + e^x} = \frac{1}{2}.$$

Thus, the limit is 0.5.

Final Answer:

$$\lim_{x\to 0} \frac{1(e^x - 1) - 1(x)}{x\,(e^x - 1)} = 0.5.$$

3. [3 marks] Find the local minima and maxima of the function $y(x) = 2x^3 - 9x^2 + 12x - 3$, using the Second Derivative Test to classify your critical points. Use this information to show that the equation $y(x) = 0$ has precisely one real root. Use the solution box only for your critical points, and be neat in your development.

Begin by differentiating the function twice (once for critical points, once for the Second Derivative Test):

$$y(x) = 2x^3 - 9x^2 + 12x - 3$$
$$y'(x) = 6x^2 - 18x + 12$$
$$y''(x) = 12x - 18$$

then set the first derivative equal to zero and solve:

$$y'(x) = 0 = 6x^2 - 18x + 12 = 6\left[x^2 - 3x + 2\right]$$

which can be factored as $0 = (x-2)(x-1)$, giving roots (critical points) $x = 1$ and $x = 2$. Substituting these into the Second Derivative function gives:

$$y''(1) = 12(1) - 18 = -6 < 0$$
$$y''(2) = 12(2) - 18 = 6 > 0$$

so the critical point at $x = 1$ is a local maximum, and the critical point at $x = 2$ is a local minimum. Now, we are asked to show that these findings imply that the equation $y(x) = 0$ has only one real root; this can also be interpreted as implying that the cubic polynomial $y(x)$ only crosses the x-axis once. If we evaluate the critical points for $y(x)$ we obtain:

$$y(1) = 2(1)^3 - 9(1)^2 + 12(1) - 3 = 2 - 9 + 12 - 3 = 2$$
$$y(2) = 2(2)^3 - 9(2)^2 + 12(2) - 3 = 16 - 36 + 24 - 3 = 1.$$

Using the Intermediate Value Theorem, this tells us that between the maximum at $x = 1$ and the minimum at $x = 2$, the function $y(x)$ cannot cross the x-axis (if it did, we would have another critical point between these two). The Second Derivative Test at $x = 2$ was positive, which says that the slopes go from negative to positive at this point; since there are no more critical points for $x > 2$ this says that the slopes will always be positive to the right of $x = 2$. Thus, the function cannot go back down lower than $y(2) = 1$, so it remains always positive.

Finally, to the left of $x = 1$ the slopes are positive (since the Second Derivative Test is negative), and there are no additional critical points, so the function $y(x)$ decreases for $x < 1$, crossing the $x-$axis to the left of $x = 1$, the only such crossing, and thus $y(x)$ has only one real root.

It was also acceptable for you to present a carefully drawn and labeled graph and explain how the information from the critical point findings helped you draw the curve.

Final Answer:

$x = 1$ is a local max, and $x = 2$ is a local min.

4. [2 marks] The pressure P in kilopascals, volume V in litres, and temperature T in degrees Kelvin, of a mole of an ideal gas have relationship

$$PV = 8.31 \cdot T.$$

Find the rate at which the pressure in a container of this gas is changing when the temperature is $300^\circ K$ and is increasing at a rate of $0.1^\circ K$/sec, and (at the same time) the volume is 100 litres and increasing at a rate of 0.2 litres/second.

Begin by rearranging the equation to form:

$$P = \frac{8.31T}{V}$$

and then differentiate implicitly:

$$P' = \frac{8.31T'V - 8.31TV'}{V^2}$$

where we have used the Quotient Rule. Substituting the values we are given ($V = 100$, $V' = 0.2$, $T = 300$, $T' = 0.1$) we obtain

$$P' = \frac{8.31(0.1)(100) - 8.31(300)(0.2)}{(100)^2} = -0.04155.$$

This is the rate at which the pressure is changing.

Final Answer:

The pressure is decreasing at a rate of 0.04155 kPa/s.

5. [2 marks] A drinking cup is to be manufactured in the shape of an upright cylinder (with no top, so you can drink from it). If the volume is fixed (but not given), we wish to use the minimum possible material to make the sides and bottom of the cup. Under this constraint, what will the ratio of the height of the cup be to the diameter of the cup?

Begin by writing the fundamental optimizing equation:

$$SA = \pi r^2 + 2\pi r h,$$

the surface area of the cup; this is the equation because we are minimizing the material used. Note that it is only πr^2 and not $2\pi r^2$ because there is no top to the cup. We also have a constraint:

$$V = \pi r^2 h$$

which is fixed, but unknown. Leave this symbolically as V:

$$h = \frac{V}{\pi r^2}$$

and substitute into the SA equation:

$$SA = \pi r^2 + 2\pi r \frac{V}{\pi r^2} = \pi r^2 + \frac{2V}{r}.$$

Differentiate this with respect to r to obtain:

$$SA' = 2\pi r - \frac{2V}{r^2} = 0 \qquad \Rightarrow \qquad 2\pi r = \frac{2V}{r^2}$$

so that $\pi r^3 = V$, or $r = \sqrt[3]{V/\pi}$. We prove this is a minimum by using the Second Derivative Test:

$$SA''\left(\sqrt[3]{V/\pi}\right) = \left[2\pi + \frac{4V}{r^3}\right]_{r=\sqrt[3]{V/\pi}} = 2\pi + \frac{4V}{V/\pi} = 6\pi > 0.$$

Thus, this is a local minimum; as there is only one critical point, it must also be a global minimum. Now, find the ratio of h to $d = 2r$:

$$\frac{h}{2r} = \frac{\frac{V}{\pi r^2}}{2r} = \frac{V}{2\pi r^3}$$

and substitute the critical point we found:

$$\frac{h}{2r} = \frac{V}{2\pi\left(\sqrt[3]{V/\pi}\right)^3} = \frac{V}{2\pi\frac{V}{\pi}} = \frac{V\pi}{2V\pi} = \frac{1}{2}.$$

Final Answer:

The ratio of the height to the diameter is $1:2$, or $1/2$.

6. [2 marks] The cost of operating a truck is $30 + 1.04 \cdot 10^{-3}v^2$ cents/km, when operating at v km/hr. If a truck driver earns $\$20$/hour, what is the cheapest speed at which to operate over a 1000km trip.

The key to this problem is unit analysis. You must make sure that your two costs (which add together to constrain the speed) are in the same units. You could use any of four options: cents/km, dollars/km, cents, or dollars. We will demonstrate this solution using dollars.

$$C(v) = \left[30 + 1.04 \cdot 10^{-3}v^2\right] \cdot 1000 \cdot \frac{1}{100} + [20] \cdot \frac{1000}{v}$$

where we have multiplied the truck operating costs by the number of km (converting it into cents), and then divided it by 100 to convert it to dollars. The driver's salary has been converted from dollars/hour to dollars by multiplying by the number of hours the trip will take, which was further computed by taking the trip length and dividing by the speed, using the physics relation $d = vt$. Now, simplify and differentiate:

$$C(v) = 300 + 0.0104v^2 + \frac{20,000}{v}$$

so

$$C'(v) = 0.0208v - \frac{20,000}{v^2} = 0 \qquad \Rightarrow \qquad 0.0208v^3 = 20,000$$

which has solution $v^3 = 961,538.46$, or $v = 98.7$ km/hr. To check that this is actually a minimum (the cheapest) speed at which to operate, use the Second Derivative Test:

$$C''(v) = 0.0208 + \frac{40,000}{v^3} \qquad \Rightarrow \qquad C''(98.7) = 0.062 > 0$$

so this is a local minimum. Since there is only one critical point on the domain, and it is a minimum, it must also be an absolute minimum, so this is the cheapest possible speed.

Final Answer:

The cheapest speed at which to operate is $v = 98.7$ km/hr.

7. [2 marks] Solve the following indefinite integral:

$$\int x^2 \cos(x) dx.$$

Begin by choosing elements for Integration by Parts:

$$\begin{aligned} u &= x^2 & v &= \sin(x) \\ du &= 2x dx & dv &= \cos(x) dx \end{aligned}$$

which gives

$$\int x^2 \cos(x) dx = x^2 \sin(x) - \int 2x \sin(x) dx$$

and taking the second integral, we again apply Integration by Parts:

$$\begin{aligned} u &= 2x & v &= -\cos(x) \\ du &= 2dx & dv &= \sin(x) dx \end{aligned}$$

so we have

$$\begin{aligned} \int x^2 \cos(x) dx &= x^2 \sin(x) - \int 2x \sin(x) dx \\ &= x^2 \sin(x) - \left[-2x \cos(x) - \int -2 \cos(x) dx \right] \\ &= x^2 \sin(x) + 2x \cos(x) - \int 2 \cos(x) dx \\ &= x^2 \sin(x) + 2x \cos(x) - 2 \sin(x) \end{aligned}$$

As this is an indefinite integral, add $+C$:

$$\int x^2 \cos(x) dx = x^2 \sin(x) + 2x \cos(x) - 2 \sin(x) + C.$$

Final Answer:

$$\int x^2 \cos(x) dx = x^2 \sin(x) + 2x \cos(x) - 2 \sin(x) + C.$$

8. [2 marks] Solve the following definite integral:

$$\int_0^1 \frac{t^5}{\sqrt[5]{t^3+1}} dt.$$

This is quite a tricky integral. If you didn't see how to do it, start by trying to use Substitution. Let $b = t^3 + 1$, which gives

$$\frac{db}{dt} = 3t^2$$

or $dt = db/3t^2$. Substituting this gives

$$\int_{t=0}^{t=1} \frac{t^5}{\sqrt[5]{b}} \frac{db}{3t^2} = \int_{t=0}^{t=1} \frac{t^3}{3\sqrt[5]{b}} db.$$

As we were not able to fully cancel the t terms, we cannot use this Substitution as it stands. You **could** proceed to do a second substitution, letting $c = b - 1$, which would then simplify to a reasonably easy integral. This is one way. Below, we do the Integration by Parts method.

Using the information from our attempted Substitution above, let $u = t^3$ (the left-over bits), and dv the rest:

$$\begin{aligned} u &= t^3 & v &= \tfrac{5}{12}\left(t^3+1\right)^{4/5} \\ du &= 3t^2 dt & dv &= t^2\left(t^3+1\right)^{-1/5} dt \end{aligned}$$

where we find v by doing Substitution integration like above, but this time it works! Now, write

$$\int_{t=1}^{t=1} \frac{t^5}{\sqrt[5]{t^3+1}} dt = \frac{5t^3}{12}\left(t^3+1\right)^{4/5}\Big|_{t=0}^{t=1} - \frac{5}{12}\int_0^1 3t^2\left(t^3+1\right)^{4/5} dt$$

Apply Substitution to this second integral (it is basically the same as the dv integral above), getting

$$\begin{aligned} \int_{t=1}^{t=1} \frac{t^5}{\sqrt[5]{t^3+1}} dt &= \frac{5t^3}{12}\left(t^3+1\right)^{4/5}\Big|_{t=0}^{t=1} - \frac{5}{12}\left[\frac{5}{9}\left(t^3+1\right)^{9/5}\right]_{t=0}^{t=1} \\ &= \frac{5}{12}t^3\left(t^3+1\right)^{4/5}\Big|_0^1 - \frac{25}{108}\left(t^3+1\right)^{9/5}\Big|_0^1 \\ &= \frac{5}{12}\left[1(2)^{4/5} - 0\right] - \frac{25}{108}\left[(2)^{9/5} - 1^{9/5}\right] \\ &= 0.15087. \end{aligned}$$

Final Answer:

$$\int_{t=1}^{t=1} \frac{t^5}{\sqrt[5]{t^3+1}} dt = 0.15087.$$

9. [2 marks] Solve the following indefinite integral:

$$\int x^3 \cos(2x)dx.$$

This problem is very similar to #7. Set it up in the same way:

$$\begin{aligned} u &= x^3 & v &= \tfrac{1}{2}\sin(2x) \\ du &= 3x^2dx & dv &= \cos(2x)dx \end{aligned}$$

so

$$\int x^3\cos(2x)dx = \frac{x^3}{2}\sin(2x) - \int \frac{3}{2}x^2\sin(2x)dx.$$

Apply Integration by Parts again:

$$\begin{aligned} u &= \tfrac{3}{2}x^2 & v &= -\tfrac{1}{2}\cos(2x) \\ du &= 3xdx & dv &= \sin(2x)dx \end{aligned}$$

so

$$\int x^3\cos(2x)dx = \frac{x^3}{2}\sin(2x) - \left[-\frac{3}{4}x^2\cos(2x) + \int \frac{3}{2}x\cos(2x)dx\right]$$

and then one last time:

$$\begin{aligned} u &= \tfrac{3}{2}x & v &= \tfrac{1}{2}\sin(2x) \\ du &= \tfrac{3}{2}dx & dv &= \cos(2x)dx \end{aligned}$$

so

$$\begin{aligned} \int x^3\cos(2x)dx &= \frac{x^3}{2}\sin(2x) - \left[-\frac{3}{4}x^2\cos(2x) + \left(\frac{3}{4}x\sin(2x) - \int \frac{3}{4}\sin(2x)\right)\right] \\ &= \frac{x^3}{2}\sin(2x) + \frac{3}{4}x^2\cos(2x) - \frac{3}{4}x\sin(2x) - \frac{3}{8}\cos(2x) + C \\ &= \frac{x\sin(2x)}{2}\left[x^2 - \frac{3}{2}\right] + \frac{\cos(2x)}{4}\left[3x^2 - \frac{3}{2}\right] + C \end{aligned}$$

Final Answer:

$$\int x^3\cos(2x)dx = \frac{x\sin(2x)}{2}\left[x^2 - \frac{3}{2}\right] + \frac{\cos(2x)}{4}\left[3x^2 - \frac{3}{2}\right] + C$$

10. [2 marks] Solve the following definite integral:

$$\int_0^{\pi/4} \frac{\cos^3(x)}{\sin^3(x)} dx.$$

There are a number of ways to solve this integral, most involving trigonometric substitutions to split the integral into different pieces. We will show the solution that comes from using only basic identities and Integration by Parts.

$$\begin{aligned} u &= \cos^2(x) & v &= -\tfrac{1}{2}\sin^{-2}(x) \\ du &= -2\cos(x)\sin(x)dx & dv &= \frac{\cos(x)}{\sin^3(x)}dx = \cos(x)\sin^{-3}(x)dx \end{aligned}$$

where we found v by using the substitution $b = \sin(x)$ so that $dv = b^{-3}db$. Then

$$\begin{aligned} \int_0^{\pi/4} \frac{\cos^3(x)}{\sin^3(x)} dx &= -\frac{1}{2}\frac{\cos^2(x)}{\sin^2(x)}\bigg|_0^{\pi/4} - \int_0^{\pi/4} \frac{2}{2}\frac{\cos(x)\sin(x)}{\sin^2(x)} dx \\ &= -\frac{1}{2}\cot^2(x)\bigg|_0^{\pi/4} - \int_0^{\pi/4} \cos(x)\sin^{-1}(x)dx \end{aligned}$$

and then use a final substitution, $b = \sin(x)$ which gives $dx = \dfrac{db}{\cos(x)}$, so

$$\int_{x=0}^{x=\pi/4} \frac{1}{b} db = \ln|b|\bigg|_{x=0}^{x=\pi/4} = \ln|\sin(x)|\bigg|_{x=0}^{x=\pi/4}$$

so

$$\begin{aligned} \int_0^{\pi/4} \frac{\cos^3(x)}{\sin^3(x)} dx &= -\frac{1}{2}\cot^2(x)\bigg|_0^{\pi/4} - \ln|\sin(x)|\bigg|_0^{\pi/4} \\ &= -\frac{1}{2}\cot^2(\pi/4) + \frac{1}{2}\cot^2(0) - [\ln|\sin(\pi/4)| - \ln|\sin(0)|] \end{aligned}$$

and we realize that $\cot(0) = \infty$ and $\ln|0| = \infty$. Thus, we need to re-write this final equation in limiting form:

$$\begin{aligned} &= \lim_{T\to 0}\left[-\frac{1}{2}\cot^2(\pi/4) + \frac{1}{2}\cot^2(T) - \ln|\sin(\pi/4)| + \ln|\sin(T)|\right] \\ &\approx \lim_{T\to 0}\left[-\frac{1}{2}\cot^2(\pi/4) + \frac{1}{2}\frac{1}{T} - \ln|\sin(\pi/4)| + \ln|T|\right] \\ &= \infty \end{aligned}$$

Note that even thought $\ln|0| = -\infty$, $\cot^2(0)$ goes to $+\infty$ faster than $\ln|0|$ goes to $-\infty$, so the solution is $+\infty$ rather than $-\infty$.

Final Answer:

The integral diverges.